Chapter I. General Concepts and Chebyshev's Inequality.

1.1. <u>Basic Ideas</u>. One of the most important facts in mathematics and in the theory of inequalities in particular is the observation that the square of a real number X can never be negative: $X^2 \geq 0$, with equality only if $X = 0$. Thus, for two positive numbers a and b, we have

$$(\sqrt{a} - \sqrt{b})^2 \geq 0,$$

or

$$a - 2\sqrt{ab} + b \geq 0.$$

This may be written in the form

$$\frac{a + b}{2} \geq \sqrt{ab} \ . \tag{1.1}$$

By replacing a by a^2 and b by b^2, we obtain an alternative form,

$$\frac{a^2 + b^2}{2} \geq ab \ , \tag{1.2}$$

which clearly holds for possibly negative a and b as well. A related inequality is given in Exercise 1.1:

$$\sqrt{\frac{a^2 + b^2}{2}} \geq \frac{a + b}{2} \ . \tag{1.3}$$

Given two positive numbers a and b, the number $\frac{a + b}{2}$ is called the <u>arithmetic</u> <u>mean</u> of a and b, while the number $\sqrt{ab}$ is called the <u>geometric</u> <u>mean</u> of a and b. What (1.1) tells us is that the geometric mean can never be greater than the arithmetic mean.

In dealing with inequalities, it is always important to know when, or under what conditions, an inequality can reduce to an equality. In

case of the inequality (1.1), we have the answer from the inequality $(\sqrt{a} - \sqrt{b})^2 \geq 0$ which we used to deduce (1.1): we have that $(\sqrt{a} - \sqrt{b})^2 = 0$ if and only if $a = b$. Hence the arithmetic and geometric means of two positive numbers a and b are equal if and only if $a = b$.

Let us proceed to exploit the relationship between the arithmetic and geometric means.

Example 1. Let a,b,c be real numbers. It follows from (1.2) that

$$\left\{\begin{array}{l} a^2 + b^2 \geq 2ab \\ b^2 + c^2 \geq 2bc \\ c^2 + a^2 \geq 2ca , \end{array}\right. \tag{1.4}$$

where we observe that equality holds in the three statements in (1.4) if and only if $a = b$, $b = c$ and $c = a$, i.e., $a = b = c$. Adding these three inequalities and dividing by 2 yields

$$a^2 + b^2 + c^2 \geq ab + bc + ca , \tag{1.5}$$

with equality if and only if $a = b = c$. (For other proofs, see Exercise 1.2.)

Example 2. Let us rewrite (1.4) in the form

$$\left\{\begin{array}{l} a^2 - ab + b^2 \geq ab \\ b^2 - bc + c^2 \geq bc \\ c^2 - ca + a^2 \geq ca . \end{array}\right.$$

Now observe that $(a + b)(a^2 - ab + b^2) = a^3 + b^3$. Thus, if we multiply the three inequalities above by $a + b$, $b + c$, and $c + a$, we have

$$\left\{\begin{array}{l} a^3 + b^3 \geq ab(a + b) \\ b^3 + c^3 \geq bc(b + c) \\ c^3 + a^3 \geq ca(c + a) , \end{array}\right.$$

where we must now require that the terms a + b, b + c, and c + a be positive, or else at least one of the inequalities must be reversed. If we now add the last three inequalities and divide by 2, we obtain

$$(1.6)\qquad a^3 + b^3 + c^3 \geq ab\,\frac{a+b}{2} + bc\,\frac{b+c}{2} + ca\,\frac{c+a}{2},$$

with equality if and only if $a = b = c$.

EXERCISES

Always indicate when equality holds.

1.1. Prove (1.3).

1.2. (a) Give another proof of (1.5) by considering inequalities of the form $(2a - b - c)^2 \geq 0$.

(b) Give yet another proof of (1.5) by first assuming (without loss of generality) that $a \geq b \geq c$ and then applying (1.1) to the quantities $(a - b)$ and $(b - c)$.

1.3. Apply (1.1) to (1.6) to obtain $a^3 + b^3 + c^3 \geq (bc)^{3/2} + (ca)^{3/2} + (ab)^{3/2}$ for $a,b,c \geq 0$. Can you derive this in another way?

1.4. Prove that the right-hand side of (1.6) is at least 3abc. (<u>Hint</u>. Multiply the triple of inequalities (1.4) by c,a,b in order and then add.) Deduce that $a^3 + b^3 + c^3 \geq 3abc$, or, equivalently, that $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$, with equality if and only if $a = b = c$.

1.5. Show that $(a + b)(b + c)(c + a) \geq 8abc$ for positive a,b,c.

1.6. Prove that $\frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} \geq 6$.

1.7. Prove that $a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c)$. (Hint. Multiply the triple of inequalities (1.4) by appropriate quantities and add.)

1.8. Show that $\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} \geq a + b + c$.

1.9. Show that $a^3 + b^3 \geq \frac{(a + b)^3}{4}$, or, more elegantly, $\sqrt[3]{\frac{a^3 + b^3}{2}} \geq \frac{a + b}{2}$.

1.10. Let a,b,c,d be positive numbers.

(a) Show that $\frac{a + b + c + d}{4} \geq \sqrt{\frac{ab + bc + cd + da}{4}}$.

(b) Use (a) to show that $\frac{a + b + c + d}{4} \geq \sqrt{\frac{ab + bc + cd + da + ac + bd}{6}}$.

1.2. Further Basic Ideas. We have seen in the previous section how we can manipulate an expression which cannot be negative into a useful inequality. To develop this idea in another direction, let $\{a_1,a_2\}$, $\{b_1,b_2\}$ be two increasing sequences* of real numbers, $a_1 \leq a_2$, $b_1 \leq b_2$. Then

$$(a_1 - a_2)(b_1 - b_2) \geq 0,$$

with equality if and only if $a_1 = a_2$ or $b_1 = b_2$. Multiplying this out, we have

$$a_1b_1 + a_2b_2 \geq a_1b_2 + a_2b_1 .$$

Then if we add $a_1b_1 + a_2b_2$ to both sides, factor, and divide by 4, we obtain an instance of Chebyshev's inequality,

$$\frac{a_1b_1 + a_2b_2}{2} \geq \frac{a_1 + a_2}{2} \cdot \frac{b_1 + b_2}{2} \tag{1.7}$$

with equality if and only if $a_1 = a_2$ or $b_1 = b_2$. We shall see in the exercises below and in the next section how (1.7) may be generalized extensively. For the moment, we consider some of its consequences. Note

----

*We are using the term "sequence" for a pair $\{a_1, a_2\}$, even though it is of length 2, with a view towards later developments.

that (1.7) holds for decreasing sequences $\{a_1,a_2\}$, $\{b_1,b_2\}$ also.

Example 3. Suppose that a,b,m,n are positive numbers. Then $\{a^m,b^m\}$ and $\{a^n,b^n\}$ are either both increasing or both decreasing sequences, according as $a \le b$ or $a \ge b$. Hence (1.7) applies:

$$\frac{a^{m+n} + b^{m+n}}{2} \ge \frac{a^m + b^m}{2} \cdot \frac{a^n + b^n}{2} \tag{1.8}$$

Note that equality holds if and only if $a = b$.

Example 4. For θ in the interval $0 \le \theta \le \pi/2$, let us consider the function $\cos^4\theta + \sin^4\theta$. Suppose that we rewrite $\cos^4\theta + \sin^4\theta$ as

$$2\left(\frac{\cos^{2+2}\theta + \sin^{2+2}\theta}{2}\right)$$

in order to use (1.8) with $m = n = 2$. Then

$$2\left(\frac{\cos^{2+2}\theta + \sin^{2+2}\theta}{2}\right) \ge 2\left(\frac{\cos^2\theta + \sin^2\theta}{2}\right)\left(\frac{\cos^2\theta + \sin^2\theta}{2}\right) = \frac{1}{2}\,.$$

Therefore

$$\cos^4\theta + \sin^4\theta \ge \frac{1}{2}\,. \tag{1.9}$$

with equality if and only if $\sin\theta = \cos\theta$, i.e., $\theta = \pi/4$.

We wish to mention here an important aspect of our interest in the conditions under which equality holds in a specific inequality, and we can use (1.9) as our first example. The function $\cos^4\theta + \sin^4\theta$ is always greater than or equal to 1/2 for all θ in $0 \le \theta \le \pi/2$ (indeed, for all θ). At the point $\theta = \pi/4$, it assumes its least value, 1/2; in other words, the function assumes its minimum value at any point θ at which the equality sign prevails in (1.9). We shall see that in a large number of cases,

techniques involving inequalities will prove to be a more powerful or more convenient tool for determining maxima and minima than the calculus.

EXERCISES

1.11. By repeated application of (1.7), prove that for three positive increasing (or decreasing) sequences $\{a_1,a_2\}$, $\{b_1,b_2\}$, $\{c_1,c_2\}$,

$$\frac{a_1b_1c_1 + a_2b_2c_2}{2} \geq \frac{a_1 + a_2}{2} \cdot \frac{b_1 + b_2}{2} \cdot \frac{c_1 + c_2}{2}.$$

Pay particular attention to the equality condition. Can the restriction that the sequence be positive be removed?

1.12. Though we shall not prove it here explicitly, note that the inequality in the previous problem may be generalized again and also put in a form like (1.8). Show that for positive a,b and a positive integer n,

$$\left(\frac{a^n + b^n}{2}\right)^{\frac{1}{n}} \geq \frac{a + b}{2}.$$

1.13. For t in the interval $0 \leq t \leq 1$, find the minimum value of

$\left(\frac{1 - t^2}{1 + t^2}\right)^{12} + \left(\frac{2t}{1 + t^2}\right)^{12}$. (<u>Hint</u>. Calculate the value of

$\left(\frac{1 - t^2}{1 + t^2}\right)^{2} + \left(\frac{2t}{1 + t^2}\right)^{2}$.)

1.14. For any pair of positive numbers a and b such that $a^2 + b^2 = 1$, show that $a^{2n} + b^{2n} \geq \frac{1}{2^{n-1}}$, where n is a positive integer.

1.15. Show that $1 + \tan^8\theta > \frac{1}{8}\sec^8\theta$ for $0 < \theta < \pi/2$, except at $\theta = \pi/4$.

1.16. Show that for positive a and b, $(a + b)(a^2 + b^2)(a^3 + b^3) \le 4(a^6 + b^6)$.

1.17. Using the methods of this section, find the shortest distance from the origin to the line $x + y - 3 = 0$.

1.18. Using the methods of this section, find the shortest distance from the origin to the parabolic arc $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

1.19. Show that the inequality is reversed in (1.7) if one of the sequences is increasing while the other is decreasing, but that the equality condition remains unchanged.

1.3. Chebyshev's Inequality. Problem 1.11 indicated one possible generalization of (1.7). We shall now give another. Consider two increasing (or, alternatively, decreasing) sequences $\{a_1,a_2,a_3\}$ and $\{b_1,b_2,b_3\}$. Then as in the previous section,

$$\begin{cases} (a_1 - a_2)(b_1 - b_2) \ge 0 \\ (a_2 - a_3)(b_2 - b_3) \ge 0 \\ (a_3 - a_1)(b_3 - b_1) \ge 0\,, \end{cases}$$

or

$$\begin{cases} a_1b_1 + a_2b_2 \ge a_1b_2 + a_2b_1 \\ a_2b_2 + a_3b_3 \ge a_2b_3 + a_3b_2 \\ a_3b_3 + a_1b_1 \ge a_3b_1 + a_1b_3\,. \end{cases}$$

If we now sum these inequalities and then add $a_1b_1 + a_2b_2 + a_3b_3$ to both sides, we arrive at

$$\text{(1.10)} \qquad \frac{a_1b_1 + a_2b_2 + a_3b_3}{3} \geq \frac{a_1 + a_2 + a_3}{3} \cdot \frac{b_1 + b_2 + b_3}{3},$$

with equality if and only if $a_1 = a_2 = a_3$ or $b_1 = b_2 = b_3$. In the exercises, the reader is asked to generalize this to two sequences of n numbers $\{a_1, \cdots, a_n\}, \{b_1, \cdots, b_n\}$:

$$\text{(1.11)} \qquad \frac{a_1b_1 + a_2b_2 + \cdots + a_nb_n}{n} \geq \frac{a_1 + a_2 + \cdots + a_n}{n} \cdot \frac{b_1 + b_2 + \cdots + b_n}{n},$$

and then to more than two sequences. The inequality (1.11) is known as Chebyshev's inequality.

Example 5. Given three real numbers, label them a,b,c in increasing order. Then by (1.10),

$$\text{(1.12)} \qquad \frac{a^2 + b^2 + c^2}{3} \geq \frac{a + b + c}{3} \cdot \frac{a + b + c}{3} = \frac{(a + b + c)^2}{9}$$

or

$$3(a^2 + b^2 + c^2) \geq a^2 + b^2 + c^2 + 2(bc + ca + ab).$$

Thus

$$a^2 + b^2 + c^2 \geq bc + ca + ab,$$

giving us another proof of (1.5). (Note that equality holds if and only if $a = b = c$.)

Example 6. Using (1.10) and (1.5) gives us

$$\text{(1.13)} \qquad \frac{a^3 + b^3 + c^3}{3} \geq \frac{a^2 + b^2 + c^2}{3} \cdot \frac{a + b + c}{3}$$

$$\geq \frac{bc + ca + ab}{3} \cdot \frac{a + b + c}{3},$$

or

$$3(a^3 + b^3 + c^3) \geq (bc + ca + ab)(a + b + c),$$

with equality if and only if $a = b = c$.

Note on the summation and product notation. To make another notation more compact, we write $\sum_{k=1}^{n} a_k$ for the sum $a_1 + a_2 + \cdots + a_n$. Thus (1.11) is

$$\frac{1}{n}\sum_{k=1}^{n} a_k b_k \geq \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right).$$

Similarly, we write $\prod_{k=1}^{n} a_k$ for the product $a_1 a_2 \cdots a_n$. Thus $\prod_{k=1}^{n} k = n!$ and

$$\prod_{k=1}^{n}\left(1 + \frac{1}{k}\right) = \prod_{k=1}^{n}\frac{k+1}{k} = \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\ \cdots\ \cdot\left(\frac{n+1}{n}\right) = n + 1.$$

EXERCISES FOR CHAPTER I

1.20. (a) Use the methods of this chapter to find the shortest distance from the origin to the plane $x + y + z - 5 = 0$.

(b) Similarly, find the shortest distance from the origin to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} - \sqrt{a} = 0$.

1.21. Find the minimum value of the function

$$f(x,y) = \sin^4 x + \left(\frac{1 - y^2}{1 - y^2}\right)^4 \cos^4 x + \left(\frac{2y}{1 + y^2}\right)^4 \cos^4 x$$

on $0 \leq x \leq \pi/2$, $0 \leq y \leq 1$. What values of x and y yield the minimum?

1.22. Let $f(x) = 1 + \cos^4 x + \sin^4 x$ on $0 \leq x \leq \pi/2$. Show that the use of (1.12) yields the inequality $f(x) \geq 4/3$, and explain this apparent discrepancy with the minimum of 3/2, which follows from (1.9).

1.23. Prove (1.11). Show that the inequality is reversed if one sequence is increasing while the other is decreasing.

1.24. Generalize (1.11) to m sequences of n terms. (*Cf*. Exercise 1.11.)

1.25. Show that if m is a positive integer and $a_1, a_2, \cdots, a_n$ are positive, then

$$\left(\frac{\sum_{k=1}^{n} a_k^m}{n}\right)^{\frac{1}{m}} \geq \frac{\sum_{k=1}^{n} a_k}{n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$. This generalizes Exercise 1.12.

1.26. Show that $(x^m + y^m)^n < (x^n + y^n)^m$ if $m > n$, where m,n are positive integers and x,y are positive numbers.

1.27. Let $x_1, \cdots, x_n$ be positive numbers such that $x_1 \cdots x_n = 1$. Show that $(1 + x_1) \cdots (1 + x_n) \geq 2^n$.

1.28. Let $x_1, \cdots, x_n$ be positive numbers such that $x_1 \cdots x_n = 1$, and let $x_{p_1}, \cdots, x_{p_n}$ be a permutation of $x_1, \cdots, x_n$. Show that $(x_1 + x_{p_1}) \cdots (x_1 + x_{p_n}) \geq 2^n$.

1.29. (a) Consider all positive x,y whose sum x + y is constant. Show that the product xy is a maximum when x = y.

(b) Consider all positive x,y whose product xy is constant. Show that the sum x + y is a minimum when x = y.

(c) If the sum of three positive numbers is constant, show their product is a maximum when the numbers are equal.

1.30. Find the maximum value of $x\sqrt{16 - x^2}$ for $0 \leq x \leq 4$.

1.31. Find the minimum value of xyz if $x + 2y + 3z = 6$, $x,y,z > 0$.

1.32. Among all rectangular parallelepipeds such that the sum of the edges is the same, show that the cube has maximum volume.

1.33. Among all rectangular parallelepipeds having the same surface area, show that the cube has the maximum volume.

1.34. Find the maximum value of $\sqrt{x} + \sqrt{y} + \sqrt{z}$ if $x^2 + y^2 + z^2 = 1$.

1.35. For n positive numbers $a_1, \cdots, a_n$, show that

$$(a_1 + \cdots + a_n)\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right) \geq n^2.$$

1.36. Let a and be be positive numbers such that $a + b = 1$. Show that

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2},$$ with equality occurring when $a = b = \frac{1}{2}$.

1.37. Let $a_1, a_2, \cdots, a_n$ be positive numbers with $\sum_{j=1}^{n} a_j = 1$. Show that

$$\sum_{j=1}^{n} \left(a_j + \frac{1}{a_j}\right)^2 \geq \frac{1}{n}(n^2 + 1)^2.$$

1.38. For positive a and b with $a + b = 1$, show that the series

$$\sum_{n=1}^{N} \frac{1}{\left(a + \frac{1}{a}\right)^n + \left(b + \frac{1}{b}\right)^n}$$ converges and that the sum does not exceed 1/3.

1.39. For $a > 1$ and $b > 1$, show that the series $\sum_{n=1}^{\infty} \frac{1}{(\log_a b + \log_b a)^n}$ converges and that the sum does not exceed 1.

1.40. Let $a_1, a_2, \cdots, a_n$ be positive and let $p > 0$, $q < 0$. Show that

$$\frac{1}{n} \sum_{i=1}^{n} a_i^{p+q} \leq \left(\frac{1}{n} \sum_{i=1}^{n} a_i^p\right)\left(\frac{1}{n} \sum_{i=1}^{n} a_i^q\right),$$ with equality if and only if

$a_1 = a_2 = \cdots = a_n$.

1.41. Let $a_1, a_2, \cdots, a_n$ be positive and p,q negative. Prove that

$$\frac{1}{n} \sum_{i=1}^{n} a_i^{p+q} \geq \left(\frac{1}{n} \sum_{i=1}^{n} a_i^p\right)\left(\frac{1}{n} \sum_{i=1}^{n} a_i^q\right).$$

1.42. Show that for positive a,b,c

$$\frac{bc}{b + c} + \frac{ca}{c + a} + \frac{ab}{a + b} \le \frac{a + b + c}{2} .$$

1.43. Prove that for a,b,c > 0

$$a^a b^b c^c \ge (abc)^{\frac{a+b+c}{3}} ,$$

with equality if and only if a = b = c.

1.44. Show that for a,b,c > 0, we have

$$\frac{a + b + c}{3} \ge \sqrt{\frac{ab + bc + ca}{3}} \ge \sqrt[3]{abc}$$

Chapter II. The Arithmetic, Geometric, and Harmonic Means

2.1. Means. A mean of two or more numbers is a function of those numbers whose value always lies between the lowest and the highest of the numbers, that is, between the extremes. In this book, we shall define and consider the arithmetic, geometric, harmonic, quadratic, power, and two symmetric means.

The arithmetic mean is commonly called the average, or, in statistics, simply the mean. It is the sum of the given numbers divided by the number of them. Thus, the arithmetic mean of the n numbers $a_1, a_2, a_3, \cdots, a_n$ is

$$\frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{1}{n}\sum_{i=1}^{n} a_i .$$

In an arithmetic progression, each term is the arithmetic mean of the terms immediately preceding and immediately following, since three consecutive terms can be written in the form $a - d$, a, $a + d$, the mean of the first and last being

$$\frac{(a - d) + (a + d)}{2} = a.$$

The geometric mean is similar. Of n positive numbers, $a_1, a_2, \cdots, a_n$, it is

$$\sqrt[n]{a_1 a_2 \cdots a_n} = \left(\prod_{i=1}^{n} a_i\right)^{1/n}.$$

In a geometric progression of positive terms, each term is the geometric mean of the two adjacent terms, since

$$\sqrt{\left(\frac{a}{r}\right)(ar)} = a.$$

The harmonic mean is defined as the reciprocal of the arithmetic mean of the reciprocals of the numbers. Thus, the <u>harmonic mean</u> of $a_1, a_2, \cdots, a_n$ is

$$\frac{1}{\left(\dfrac{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots + \dfrac{1}{a_n}}{n}\right)} = \frac{n}{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots + \dfrac{1}{a_n}} = \frac{n}{\sum_{i=1}^{n} \dfrac{1}{a_i}}$$

A **harmonic** progression is a progression such that the reciprocals of its terms form an arithmetic progression. Thus, each term in such a progression is the harmonic mean of the adjacent two.

We should justify the use of the term "mean", for a mean lies between the extreme values. We shall seldom consider negative numbers, so let $a_1, a_2, \cdots, a_n$ be n positive numbers such that a_1 is the smallest (it is not necessarily unique) and a_n the largest. Then

$$\begin{gathered} a_1 \le a_1 \le a_n \\ a_1 \le a_2 \le a_n \\ \vdots \\ a_1 \le a_n \le a_n . \end{gathered}$$

Adding and dividing by n, we obtain

$$a_1 \le \frac{a_1 + a_2 + \cdots + a_n}{n} \le a_n ,$$

which is what is desired. If we multiply, instead, and take the n-th root, we have

$$a_1 \le \sqrt[n]{a_1 a_2 \cdots a_n} \le a_n .$$

Furthermore, if we take reciprocals, we have

$$\frac{1}{a_1} \ge \frac{1}{a_1} \ge \frac{1}{a_n}$$

$$\frac{1}{a_1} \ge \frac{1}{a_2} \ge \frac{1}{a_n}$$

$$\vdots$$

$$\frac{1}{a_1} \ge \frac{1}{a_n} \ge \frac{1}{a_n} .$$

Adding, we get

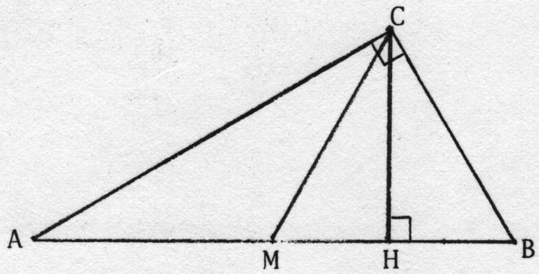
$$\frac{n}{a_1} \ge \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \ge \frac{n}{a_n}$$

or

$$a_1 \le \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \le a_n .$$

Thus, these are indeed means.

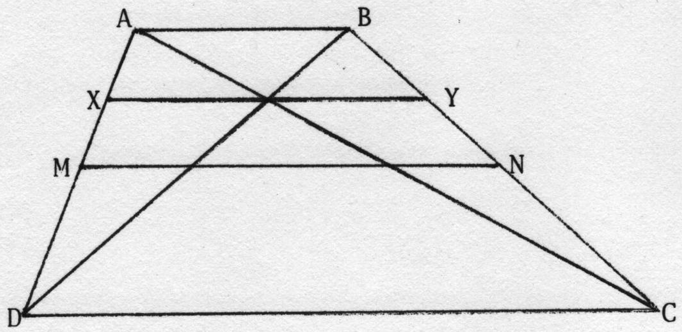
All three means appear in elementary Euclidean geometry. For example, if ΔABC has a right angle at C (see Figure 2.1), if H is the foot of the perpendicular from C to $\overline{AB}$, and if M is the mid-point of $\overline{AB}$, then CH is the geometric mean of AH and HB, and CM is the arithmetic mean of AH and HB.



$$CH = \sqrt{AH \cdot HB}, \quad CM = \frac{AH + HB}{2}.$$

Figure 2.1.

Also, given a trapezoid ABCD (see Figure 2.2), with $\overline{AB}$ parallel to $\overline{CD}$, if M and N are the mid-points of $\overline{AD}$ and $\overline{BC}$, and if X and Y are points on $\overline{AD}$ and $\overline{BC}$ such that $\overline{XY}$ is parallel to $\overline{AB}$ through the intersection of the diagonals, $\overline{AC}$ and $\overline{BD}$, then MN is the arithmetic mean of AB and CD, while XY is their harmonic mean.



$$MN = \frac{AB + CD}{2},$$

$$\frac{2}{XY} = \frac{1}{AB} + \frac{1}{CD}.$$

Figure 2.2.

2.2. <u>Comparison of the Arithmetic and Geometric Means</u>. Given these means, we may ask if there is any relation between them. Figure 2.1 suggests that, at least for two numbers, the arithmetic mean is at least the geometric mean; in other words, for any two positive numbers, a and b,

$$\frac{a + b}{2} \geq \sqrt{ab}, \tag{2.1}$$

with equality if and only if $a = b$. Indeed, this was shown in Chapter 1 as inequality (1.1). In Problem 1.24, we saw also that for any three positive numbers, a, b, and c,

$$\frac{a + b + c}{3} \geq \sqrt[3]{abc},$$

with equality if and only if $a = b = c$. In fact, this relation is true in general for n positive numbers:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}, \tag{2.2}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$. This is known as the theorem of the arithmetic and geometric means. This inequality may be proved in many ways, and we present first one of the simplest proofs.

To prove (2.2), we proceed by induction to prove the following assertion:

> *If the product of n positive numbers is equal to 1, then the sum cannot be less than n.*

From what we have done in Chapter 1, we know that the assertion is true for $n = 2$ since, if $a_1 a_2 = 1$, or $a_2 = 1/a_1$, we have $a_1 + 1/a_1 \geq 2$. Suppose next that we assume that the assertion is true for $1, 2, \cdots, k$; we wish to prove that, if $a_1 a_2 \cdots a_k a_{k+1} = 1$, then $a_1 + a_2 + \cdots + a_k + a_{k+1} \geq k + 1$, with equality only when $a_1 = a_2 = \cdots = a_{k+1}$. We may assume that at least two of the numbers-- --we call them here a_1 and a_{k+1}--have the property that $a_1 < 1$ and $a_{k+1} > 1$, for if all the a_i are less than 1, for example, their product could not be 1:

$$(a_1 a_{k+1}) a_2 \cdots a_k = 1.$$

If we set $z = a_1 a_{k+1}$, we have that the product of the k numbers $z, a_2, \cdots, a_k$ is 1, so that, by the induction hypothesis, their sum cannot be less than k:

$$z + a_2 + \cdots + a_k \geq k.$$

However, we observe that

$$a_1 + a_2 + \cdots + a_k + a_{k+1} =$$

$$= (z + a_2 + \cdots + a_k) + a_{k+1} + a_1 - z \geq$$

$$\geq k + 1 + a_{k+1} + a_1 - z - 1 = k + 1 + a_{k+1} + a_1 - a_1 a_{k+1} - 1$$

$$= k + 1 + (a_{k+1} - 1)(1 - a_1),$$

and this last expression is greater than k + 1 since $(a_{k+1} - 1)(1 - a_1) > 0$. This shows that

$$a_1 + a_2 + \cdots a_k + a_{k+1} \geq k + 1,$$

and the assertion is completely proved. We remark that the sum $\sum_{i=1}^{k+1} a_i$ is actually equal to k + 1 only if $a_1 = a_2 = \cdots = a_{k+1}$.

As an immediate corollary of the assertion above, we have that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

with equality only if $a_1 = a_2 = \cdots = a_n$. For, if we set $g = \sqrt[n]{a_1 a_2 \cdots a_n}$, we have the identity

$\sqrt[n]{\left(\frac{a_1}{g}\right)\left(\frac{a_2}{g}\right) \cdots \left(\frac{a_n}{g}\right)} = 1$, or $\left(\frac{a_1}{g}\right)\left(\frac{a_2}{g}\right) \cdots \left(\frac{a_n}{g}\right) = 1$, so that the sum $\frac{a_1}{g} + \frac{a_2}{g} + \cdots + \frac{a_n}{g}$ cannot be less than n:

$$\frac{a_1}{g} + \frac{a_2}{g} + \cdots + \frac{a_n}{g} \geq n,$$

or

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq g,$$

with equality only if $\frac{a_1}{g} = \frac{a_2}{g} = \cdots = \frac{a_n}{g}$, or, what is the same thing, only if $a_1 = a_2 = \cdots = a_n$.

Let us give another proof of (2.2), which we feel gives a different insight into (2.2). By (2.1), if the sum of two numbers is given, their product is greatest when they are equal. This may suggest that, more generally, their product is greater the closer they are. That is,

Lemma. Let $x < y$ and $0 < a < y - x$. Then $xy < (x + a)(y - a)$.

Proof. We have $x + a < y$ and $a > 0$. Thus $a(x + a) < ay$, or, if we add $xy - a(x + a)$ to both sides,

$$xy < xy + ay - a(x + a) = (x + a)(y - a).$$

We notice that, in (2.2), if not all the numbers are equal, then by the definition of mean, there exist two of them, a_1 and a_2, say, such that $a_1 < A < a_2$, where $A = \frac{1}{n}\sum_{i=1}^{n} a_i$. Thus, if we let $a_1' = A$, $a_2' = a_1 + a_2 - A$, we have $a_1 + a_2 = a_1' + a_2'$ and, by the Lemma above, $a_1a_2 < a_1'a_2'$, or

$$\sqrt[n]{a_1 a_2 a_3 \cdots a_n} < \sqrt[n]{a_1' a_2' a_3 \cdots a_n} .$$

Repeating this process at most n - 2 times, we arrive at a set of n numbers all equal to A, whose geometric mean, namely A, is greater than the geometric mean of the original numbers, namely $\sqrt[n]{a_1 a_2 \cdots a_n}$. That is to say,

$$\sqrt[n]{a_1 a_2 \cdots a_n} < A = \frac{a_1 + a_2 + \cdots + a_n}{n} ,$$

unless $a_1 = a_2 = \cdots = a_n$, in which case we have equality. Thus, (2.2) is established.

Note that (2.2) implies that when the product of n numbers is given, their sum is a minimum when they are all equal, or, what is the same thing, when the sum of n numbers is given, their product is a maximum when they are all equal.

Example 1. Prove that $n! < (\frac{n+1}{2})^n$ for $n > 1$.

This is a simple consequence of the theorem of arithmetic and geometric means, for

$$\begin{aligned} n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n &< \left(\frac{1 + 2 + \cdots + n}{n}\right)^n \\ &= \left(\frac{n(n+1)/2}{n}\right)^n = \left(\frac{n+1}{2}\right)^n . \end{aligned}$$

Note that equality cannot hold because the n numbers are not equal for $n > 1$.

Example 2. A simple consequence of (2.1) is that

$$ax + \frac{b}{x} \geq 2\sqrt{ab} , \qquad (2.3)$$

with equality if and only if $x = \sqrt{\frac{b}{a}}$. Thus, suppose we are required to minimize

$$f(x) = \frac{(a + x)(b + x)}{c + x},$$

where a, b, c are constants. To use (2.3), let us substitute

$$y = c + x.$$

Then

$$\begin{aligned} f(x) &= \frac{(a - c + y)(b - c + y)}{y} \\ &= \frac{(a - c)(b - c)}{y} + y + (a + b - 2c) \\ (*) \qquad &\geq 2\sqrt{(a - c)(b - c)} + (a + b - 2c) = (\sqrt{a - c} + \sqrt{b - c})^2, \end{aligned}$$

with equality if and only if $y = \sqrt{(a - c)(b - c)}$. That is, f(x) has its minimum at $x = \sqrt{(a - c)(b - c)} - c$, where $f = (\sqrt{a - c} - \sqrt{b - c})^2$. If this value of x is complex, then, since equality in (*) can never occur, f has no minimum in the real domain.

<u>Example 3</u>. Given a triangle with sides of lengths a, b and c, Heron's formula gives the area of the triangle, K, as

$$K = \sqrt{s(s - a)(s - b)(s - c)},$$

where s, the semi-perimeter, equals $\frac{a + b + c}{2}$. Let us use this and (2.2) to find the triangle of given perimeter which has the most area. We have

$$K^2 = s(s - a)(s - b)(s - c)$$

$$\leq s\left(\frac{s - a + s - b + s - c}{3}\right)^3 = s\left(\frac{3s - 2s}{3}\right)^3$$

$$= \frac{s^4}{27},$$

with equality if and only if $s - a = s - b = s - c$, or $a = b = c$. Thus, the equilateral triangle has the most area, namely

$$K = \sqrt{\frac{s^4}{27}} = \frac{\sqrt{3}}{4}a^2.$$

The inequality just derived also shows that the equilateral triangle has the least perimeter among all triangles of a given area.

Example 4. Let us expand $(a + b + c)^2$ and use (2.1):

$$(a + b + c)^2 = (a^2 + b^2 + c^2) + (2ab + 2bc + 2ca)$$

$$\leq (a^2 + b^2 + c^2) + [(a^2 + b^2) + (b^2 + c^2) + (c^2 + a^2)]$$

$$= 3(a^2 + b^2 + c^2), \quad \text{or}$$

(2.4) $$\frac{a + b + c}{3} \leq \sqrt{\frac{a^2 + b^2 + c^2}{3}},$$

with equality if and only if $a = b = c$.

Example 5. Maximize $y = x^2(1 - 2x)$.

If we consider y as a product of two factors, x^2 and $(1 - 2x)$, their sum is not constant, so that (2.2) cannot be used. If, however, we consider y as the product of three factors, x, x and $(1 - 2x)$, then their sum, $x + x + 1 - 2x = 1$, is constant. Hence, y will be a maximum, namely $(1/3)^3$, when $x = x = 1 - 2x$, or $x = 1/3$. This applies only to

$x > 0$, since (2.2) is applicable only to positive numbers. At $x = 1/3$, therefore, we have a relative, or local, maximum, y taking on arbitrarily large values on $x \leq 0$.

Example 6. Maximize $y = x\sqrt{1 - x^2}$.

The maximum of y occurs at the maximum of $y^2 = x^2(1 - x^2)$, that is, when $x^2 = 1 - x^2$, or $x = \frac{1}{\sqrt{2}}$, since

$$y^2 \leq \left(\frac{x^2 + 1 - x^2}{2}\right)^2 = \frac{1}{4}.$$

Thus, the maximum is at $(\frac{1}{\sqrt{2}}, \frac{1}{2})$.

Exercises

2.1. Given n positive numbers $a_1, a_2, \cdots, a_n$. Prove that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_n}{a_1} \geq n.$$

2.2. Prove the generalization of (2.4), namely

$$\frac{1}{n}\sum_{i=1}^{n} a_i \leq \sqrt{\frac{1}{n}\sum_{i=1}^{n} a_i^2} \tag{2.5}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$. The term on the right is known as the quadratic mean, or root mean square, of the n numbers.

2.3. If a, b, c, d are the sides of a quadrilateral, and ε is the sum of two opposite angles, Bretschneider's formula gives the area, K, as $K^2 = (s - a)(s - b)(s - c)(s - d) - abcd\cos^2\frac{\varepsilon}{2}$, where s is the

simiperimeter. Assuming this formula, prove that the quadrilateral of given perimeter with maximum area is the square.

2.4. Find the dimensions of the right circular cylinder of maximum volume inscribed in a right circular cone having altitude H and radius of base R.

2.5. (a) Find the right circular cone of largest volume which can be inscribed in a sphere of radius a.

(b) Find the right circular cylinder of largest volume which can be inscribed in a sphere of radius a.

2.6. Where does the maximum occur of $(a + x)^3(a - x)^4$, $|x| < a$, where a is a constant?

2.7. Prove that

$$\sqrt[3]{(a_1 + b_1)(a_2 + b_2)(a_3 + b_3)} \geq \sqrt[3]{a_1a_2a_3} + \sqrt[3]{b_1b_2b_3} .$$

(Hint: Cube, expand, and use (2.2).)

2.8. Prove that

$$a_1 a_2^2 a_3^3 a_4^4 \leq \left(\frac{a_1 + 2a_2 + 3a_3 + 4a_4}{10}\right)^{10} .$$

2.9. (a) Maximize $(1 - x)^5(1 + x)(1 + 2x)^2$.

(b) Maximize $(x + 5)^2(5x - 7)(11 - x)^9(x + 1)$, $x > 0$.

2.10. Prove that

$$\frac{a + b + c}{3} \geq \frac{\sqrt[3]{(a + b)(b + c)(c + a)}}{2} \geq \sqrt[3]{abc} .$$

2.11. Given n positive numbers $x_1, x_2, \cdots, x_n$, and define the n quantities $\sigma_1, \sigma_2, \cdots, \sigma_n$ by

$$\sigma_1 = x_1 + x_2 + \cdots + x_n = \sum_{i=1}^{n} x_i,$$

$$\sigma_2 = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n = \sum_{i<j} x_ix_j,$$

$$\cdots$$

$$\sigma_m = x_1x_2\cdots x_m + \cdots = \sum_{i_1<i_2<\cdots<i_m} x_{i_1}x_{i_2}\cdots x_{i_m},$$

$$\cdots$$

$$\sigma_n = x_1x_2\cdots x_n = \prod_{i=1}^{n} x_i.$$

(The quantities $\sigma_1, \cdots, \sigma_n$ are called the <u>elementary symmetric functions on</u> $x_1, \cdots, x_n$.) Show that

$$1 + \sum_{i=1}^{n} \sigma_i = \prod_{i=1}^{n} (1 + x_i) \geq (1 + \sqrt[n]{\sigma_n})^n$$

by showing that

$$\sigma_m \geq \binom{n}{m} \sigma_n^{m/n}, \qquad m = 1, 2, \cdots, n.$$

(Notice that this gives an alternative solution of Problem 1.22 when $\sigma_n = 1$.)

2.12. (a) Prove that, if the product of two positive numbers is given, their sum is less, the closer they are.

(b) Give another proof of (2.2), similar to the first proof given in the text, but keeping the product of the n numbers constant while varying the sum.

2.13. Prove (2.2) by the third method of solution of Problem 2.2, i.e., by backward induction.

2.14. (a) When the perimeter of a rectangle is given, show that the square has the maximum area.

(b) Given the maximum of people to sit around a rectangular table, one side of which is against a wall, determine the shape which has the most table area.

2.15. (a) Given the volume of a box. Prove that its surface area is a minimum when it is a cube.

(b) Given the surface area of a box without a top. When is its volume a maximum?

2.16. Construct with 96 square inches of siding a rectangular box of maximum volume, open at the top, with two vertical partitions inside the box which divide it into three smaller boxes, each open at the top.

2.17. (a) Show that the arithmetic mean of the squares of the first n integers is greater than $(n + 1)^2/4$, $n > 1$.

(b) Show that the arithmetic mean of the cubes of the first n integers is greater than $(n + 1)^3/8$, $n > 1$.

2.18. Show that for any positive integer k, the arithmetic mean of the k-th powers of the first n integers is greater than $(n + 1)^k/2^k$, $n > 1$.

2.19. Let the perimeter of a sector of a circle be given. Find its maximum area.

2.20. Prove that

$$(x + y)^2 + (y + z)^2 + (z + x)^2 \geq 4\sqrt{3}\,\sqrt{(x + y + z)xyz}\ .$$

2.21. Let a triangle have sides a, b, c. Let s be the semiperimeter, K the area, R the radius of the circle through its vertices, and r the radius of the inscribed circle. Given that $K = rs$ and $R = \frac{abc}{4K}$, prove that $2r \leq R$. When does equality hold?

2.22. Prove that, for any <u>real</u> numbers $a_1, \cdots, a_n$,

$$\sqrt{a_1^2 + (1 - a_2)^2} + \sqrt{a_2^2 + (1 - a_3)^2} + \cdots + \sqrt{a_n^2 + (1 - a_1)^2} \geq n\frac{\sqrt{2}}{2}.$$

Give conditions for equality.

2.23. Let a and b be positive constants. For any $\varepsilon > 0$, find the minimum value of $\varepsilon ax + \frac{1}{\varepsilon}\frac{b}{x}$.

2.3. <u>The Harmonic Mean</u>. Figure 2.2 suggests that the harmonic mean is less than or equal to the arithmetic mean. This is true, in fact, and, moreover, it is not greater than the geometric mean. This is quite easy to prove, for it follows directly from the theorem of arithmetic and geometric means. We have

$$\frac{\sum_{i=1}^{n} \frac{1}{a_i}}{n} \geq \left(\prod_{i=1}^{n} \frac{1}{a_i}\right)^{1/n},$$

or

$$\frac{n}{\sum_{1}^{n} \frac{1}{a_i}} \leq \left(\prod_{i=1}^{n} a_i\right)^{1/n} \leq \frac{\sum_{1}^{n} a_i}{n}, \tag{2.6}$$

where $a_i > 0$ for all i and equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Example 7. (2.6) implies that

$$\left(\sum_1^n a_i\right)\left(\sum_1^n \frac{1}{a_i}\right) \geq n^2.$$

This can also be proved without (2.6) in a manner similar to Example 4. Let us expand the left side, getting

$$\sum_{i=1}^{n} a_i \frac{1}{a_i} + \sum_{\substack{i<j \\ i,j=1,\cdots,n}} \left(\frac{a_i}{a_j} + \frac{a_j}{a_i}\right),$$

where the index on the second summation sign means that we are to sum over all i,j from 1 to n such that $i < j$. (For example, if $n = 3$, then

$$\sum_{\substack{i<j \\ i,j=1,2,3}} \left(\frac{a_i}{a_j} + \frac{a_j}{a_i}\right) = \left(\frac{a_1}{a_2} + \frac{a_2}{a_1}\right) + \left(\frac{a_1}{a_3} + \frac{a_3}{a_1}\right) + \left(\frac{a_2}{a_3} + \frac{a_3}{a_2}\right) .)$$

Of course, $a_i \cdot \frac{1}{a_i} = 1$ and $\frac{a_i}{a_j} + \frac{a_j}{a_i} \geq 2$, with equality if and only if $a_i = a_j$. The second summation has $\binom{n}{2} = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2}$ terms (there are n choices for i and $n - 1$ for j, after i has been picked, but because of the requirement that $i < j$, there are $\frac{1}{2}n(n - 1)$ choices for (i,j)). Thus, the expansion is at least

$$n + \frac{n(n-1)}{2} \cdot 2 = n^2,$$

and can only be n^2 when all the a_i's are equal.

Example 8. Minimize $y = \frac{1}{x} + \frac{1}{1 - 4x}$, $0 < x < \frac{1}{4}$. We have

$$y = \frac{1}{2x} + \frac{1}{2x} + \frac{1}{1 - 4x} \geq \frac{3^2}{2x + 2x + 1 - 4x} = 9,$$ with equality if and only if $2x = 1 - 4x$, or $x = \frac{1}{6}$.

Exercises

2.24. If $x + y + z = 1$, prove that

$$(1 + \frac{1}{x})(1 + \frac{1}{y})(1 + \frac{1}{z}) \geq 64.$$

2.25. Minimize $y = \frac{1}{1 - x} + \frac{1}{1 + x} + \frac{2}{1 + x^2}$ on $|x| < 1$.

2.26. Prove that the harmonic mean of two positive numbers, given their sum, is greater the closer they are.

2.27. Minimize $y = \frac{5}{1 - x} + \frac{1}{1 + x} + \frac{2}{1 + 2x}$ on $-\frac{1}{2} < x < 1$.

2.28. Find the right circular cone of minimum volume circumscribed about a right circular cylinder of height h and radius of base r.

2.4. Undetermined Coefficients. When we are to maximize a function such as $y = x(1 - 2x)$, we cannot apply (2.2) as the function stands, but we must make the simple observation that $y = \frac{1}{2}[(2x)(1 - 2x)]$, whence we can apply (2.2) to the quantity inside the brackets. The point of introducing the coefficient of 2 to the factor x is that the factors now sum to a constant independent of x. Sometimes the coefficient cannot be found at sight, however.

Example 9. A Norman window is a window consisting of a rectangle surmounted by a semicircle. Find the relative dimensions of the Norman window of fixed perimeter admitting the greatest amount of light.

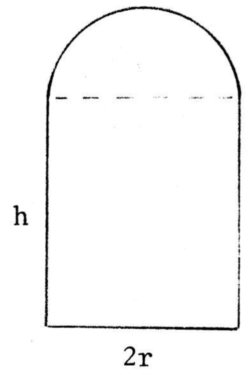


Figure 2.3.

If h is the altitude of the rectangle and r the radius of the semicircle, then the area of the window is

$$(*) \qquad A = 2rh + \frac{\pi r^2}{2} = \frac{r}{2}(\pi r + 4h),$$

which is proportional to the amount of light admitted. The perimeter is

$$(**) \qquad 2r + \pi r + 2h = C,$$

where C is a constant. We desire to multiply the first factor in $(*)$ by some positive constant coefficient so that the sum of the factors in $(*)$ is constant. Assume that the coefficient exists and call it α. We have

$$A = \frac{1}{2\alpha}\,[(\alpha r)(\pi r + 4h)]$$

and, from (xx),

$$4r + 2\pi r + 4h = 2C.$$

α must satisfy the condition we desire, that is,

$$\alpha r + \pi r + 4h = 4r + 2\pi r + 4h (= 2C).$$

This gives $\alpha = 4 + \pi$ and

$$A \leq \frac{1}{2(4+\pi)}\left[\frac{(4+\pi)r + (\pi r + 4h)}{2}\right]^2 = \frac{C^2}{2(4+\pi)},$$

where A attains this maximum value if and only if $(4 + \pi)r = \pi r + 4h$, or $r = h$.

This is the simplest application of the method of undetermined coefficients, a method which is useful in many areas of mathematics. When there are three or more factors, however, there are more considerations.

Example 10. One way to maximize the function $y = x(a^2 - x^2)$, where a is constant, is to square and use (2.2):

$$y^2 = x^2(a^2 - x^2)^2 = \tfrac{1}{2}(2x^2)(a^2 - x^2)(a^2 - x^2)$$
$$\leq \frac{1}{2}\left(\frac{2a^2}{3}\right)^3,$$

with equality at $2x^2 = a^2 - x^2$, or $x = \frac{a}{\sqrt{3}}$.

However, we need not square y. We can use the method of undetermined coefficients. We have

$$(\ast) \qquad y = x(a - x)(a + x) = \frac{1}{\alpha\beta}[\alpha x][\beta(a - x)][a + x],$$

where α and β are to be determined. We desire that the sum of the factors in $(\ast)$ be constant, or, looking at just the coefficient of x in the sum,

$$(\ast\ast) \qquad \alpha - \beta + 1 = 0.$$

We must have, also, that we can solve for x, since we have three factors which must be equal for some value of x, if we are to apply (2.2). That these simultaneous equations have a solution imposes another condition on α and β, namely, that the solution for x found by equating any pair always be the same. Thus, we must have identical solutions to the equations

$$\alpha x = a + x,$$
$$\beta(a - x) = a + x \quad ,$$

the solution of the first being

$$(\ast\ast\ast) \qquad x = \frac{a}{\alpha - 1}$$

and of the second

$$x = \frac{\beta - 1}{\beta + 1} a = \frac{\alpha}{\alpha + 2} a \ ,$$

where we have substituted for β from $(\ast\ast)$. That is to say,

$$\frac{a}{\alpha - 1} = \frac{\alpha}{\alpha + 2} a \ ,$$

$$\alpha^2 - 2\alpha - 2 = 0 \ ,$$

$$\text{or} \qquad \alpha = 1 + \sqrt{3} \ ,$$

since α must be positive. The coefficient $\beta = \alpha + 1$ is also positive. Using (***), this gives

$$x = \frac{a}{\sqrt{3}},$$

in agreement with the first solution.

Note that in (*), it would have been useless to put in one coefficient for each factor. Our two requirements--that the sum of the factors be constant and that there be a common solution for x--depend only on the ratio of the coefficients. Thus, if we had three undetermined coefficients, we could divide them all by the last one, say, leaving two undetermined and one equal to unity.

The reason why we must solve two equations in two variables, which reduces to solving a quadratic, is that there are three different factors in y. In the calculus, we set the derivative of the function to be maximized to zero and solve the resulting equation. Here, we have a cubic for y, and hence **the calculus gives us a quadratic to solve.**

Exercises

2.29. Maximize $y = x^m(a^2 - x^2)^n$, where a is constant and m,n are positive integers.

2.30. Maximize $x(a^2 - x^2)(2a - x)$, where a is constant.

2.31. (a) In a given segment of a circle, inscribe the rectangle of greatest area. (A segment is the portion between a chord and its corresponding minor arc.)

(b) In a given segment of an ellipse, whose bounding chord is perpendicular to a principal axis, inscribe the rectangle of greatest area.

2.32. (a) In a given segment of a sphere, inscribe the box of greatest volume.

(b) In a given segment of an ellipsoid, whose bounding plane is perpendicular to a principal axis, inscribe the box of greatest volume.

2.33. Given the paraboloid $z + x^2 + y^2 = C > 0$, inscribe, above the x-y plane, (a) the cylinder and (b) the box of maximum surface area.

2.34. Inscribe the rectangle of greatest area under each of the curves $y = 1/x^p$, $p = 1, 2, 3$, to the right of $x = a > 0$. (That is, its vertices will be at $(a,0)$, $(x,0)$, $(x,\frac{1}{x^p})$, $(a,\frac{1}{x^p})$.)

Exercises for Chapter II

2.35. Minimize the surface area of a right prism, given its volume and the shape of its (polygonal) base.

2.36. Given the area and one angle of a triangle, minimize

(a) the sum of the two including sides;

(b) the opposite side;

(c) the perimeter.

2.37. Maximize $z = (x + 2y)(11 - 3x - y)(2x - y + 1)$ on $x > 0$, $0 < y < 5$.

2.38. Find all positive relative maxima of $z = (x + 2y - 10)(2x + y - 26)(x + 3y - 23)$ on $x, y \geq 0$.

2.39. Prove that the arithmetic mean of the arithmetic and harmonic means of two numbers is not less than the geometric mean of those numbers.

2.40. Prove that for any positive numbers, a, b, c, such that $a + b + c = 1$,

$$\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right) \geq 8.$$

2.41. Prove that $a^3 + b^3 + c^3 + 15abc \leq 2(a + b + c)(a^2 + b^2 + c^2)$.

2.42. Prove that $(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 9a^2b^2c^2$.

2.43. Find the distance from (4,-2,1) to the plane

$2x - 3y + 6z - 13 = 0.$

2.44. Prove that if $\sum_1^n x_i = \sum_1^n y_i$, $B \geq A$, and $x_i \geq A$, $y_i \leq B$, $i = 1, \cdots, n$,

then

$$\sum_1^n \sqrt{(x_i - A)^2 + (y_i - B)^2} \geq \frac{n\sqrt{2}}{2}(B - A).$$

2.45. Given an angle and a point inside it. Pass a line through the point to cut off the triangle of minimum area.

Chapter III. The Quadratic Function and the Cauchy-Schwarz Inequality.

3.1. Introduction. In this chapter we shall investigate the use of the quadratic function $y = ax^2 + bx + c$ and the quadratic equation $ax^2 + bx + c = 0$. Our first observation is that **finding the** vertex of the parabola

$$y = ax^2 + bx + c \tag{3.1}$$

is equivalent to finding the maximum or the minimum of the function (3.1); as we shall see, we shall have a minimum or a maximum according as a is positive or negative.

Example 1. Examine the function

$$y = 3x^2 - 12x + 17$$

for a possible maximum or minimum. As in the case of analytic geometry, where we complete squares to find the coordinates of the vertex of the parabola, we have

$$y = 3(x^2 - 4x + 4) + 17 - 12,$$

or

$$y = 3(x - 2)^2 + 5.$$

The vertex of the parabola is thus at the point (2,5), and, because the coefficient of the x^2-term is positive, namely 3, the parabola opens upward. Since the vertex is the lowest point on the parabola, it follows that the function $3x^2 - 12x + 17$ assumes its minimum value 5 at the point $x = 2$.

Let us look at the solution of the problem in Example 1 from a different point of view, which can then be extended to a wider range of applicability. By completing squares, we have transformed the function $3x^2 - 12x + 17$ to the form $5 + 3(x - 2)^2$, which is the sum of a constant, 5, and a non-negative variable term $3(x - 2)^2$; hence, the function $5 + 3(x - 2)^2$ will be a minimum whenever we add to 5 the least possible amount, which, in this case, is 0, when $x = 2$.

By the same reasoning, the function $5 - 3(x - 2)^2$ will be a maximum if we subtract from 5 the least possible amount, again 0, when $x = 2$.

This point of view allows us to formulate the following principle, which no longer depends on the properties of a parabola:

> *Let A be a constant and let $F(x)$ be a non-negative function. Then the function $A + F(x)$ will be a minimum whenever $F(x)$ is a minimum, and the function $A - F(x)$ will be a maximum whenever $F(x)$ is a minimum.*

Example 2. Determine whether the function $\frac{(x + 3)(x + 5)}{x + 6}$ has any maxima or minima. If we set $x + 6 = z$, then $x = z - 6$, and the function becomes $\frac{(z - 3)(z - 1)}{z}$, or

$$\frac{z^2 - 4z + 3}{z} = z + \frac{3}{z} - 4 = z - 2\sqrt{3} + \frac{3}{z} - 4 + 2\sqrt{3}$$

$$= (\sqrt{z} - \frac{\sqrt{3}}{\sqrt{z}})^2 + 2\sqrt{3} - 4.$$

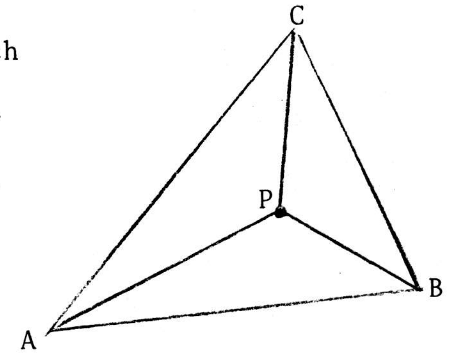
This last expression is of the form $A + F(z)$, where $A = 2\sqrt{3} - 4$ and $F(z)$ is non-negative. The expression for $F(z)$ will assume its

minimum when $\sqrt{z} - \sqrt{3}/\sqrt{z} = 0$ or when $z = \sqrt{3}$, and the minimum value of the original expression will be $2\sqrt{3} - 4$, which is achieved when $x = \sqrt{3} - 6$.

Example 3. In Exercise 1.32, we treated the function $x + 1/x$ for $x > 0$. If we write $x + \frac{1}{x} = (\sqrt{x} - 1/\sqrt{x})^2 + 2$, then, by the principle, the minimum value $A = 2$ occurs when $(\sqrt{x} - 1/\sqrt{x})^2 = 0$, or when $x = 1$.

We pose next a geometrical problem which can be solved by minimizing a quadratic function, and which has many applications.

Example 4. Given a triangle with vertices A, B, C. Find the point P such that the sum of the squares of the distances from P to A, B, C is a minimum. Let the coordinates of A, B, C be (x_1,y_1), (x_2,y_2), (x_3,y_3), respectively, and let the coordinates of P be (x,y).



Then our problem is to find x and y so that the expression

$S \equiv \overline{PA}^2 + \overline{PB}^2 + \overline{PC}^2 =$

$$(x - x_1)^2 + (y - y_1)^2 + (x - x_2)^2 + (y - y_2)^2 + (x - x_3)^2 + (y - y_3)^2 \tag{3.2}$$

is a minimum. Now S may be written as the sum of two quadratic expressions in x and y:

$$S = A_1x^2 + B_1x + C_1 + A_2y^2 + B_2y + C_2 , \tag{3.3}$$

where

$$A_1 = A_2 = 3$$

$$B_1 = -2(x_1 + x_2 + x_3), \qquad B_2 = -2(y_1 + y_2 + y_3),$$

$$C_1 = x_1^2 + x_2^2 + x_3^2 , \qquad C_2 = y_1^2 + y_2^2 + y_3^2 .$$

Since $A_1 = A_2 = 3$, and since x and y are independent, S will achieve its minimum when the two quadratic expressions are simultaneously a minimum. Thus, by completing squares, which requires only the use of A_1, B_1 and A_2, B_2, we have

$$S = 3\left[x^2 - \frac{2}{3}(x_1 + x_2 + x_3)x + \frac{(x_1 + x_2 + x_3)^2}{9}\right] + C_1 - \frac{1}{3}(x_1 + x_2 + x_3)^2$$

$$+ 3\left[y^2 - \frac{2}{3}(y_1 + y_2 + y_3)y + \frac{(y_1 + y_2 + y_3)^2}{9}\right] + C_2 - \frac{1}{3}(y_1 + y_2 + y_3)^2$$

$$= 3\left[x - \frac{x_1 + x_2 + x_3}{3}\right]^2 + 3\left[y - \frac{y_1 + y_2 + y_3}{3}\right]^2$$

$$+ C_1 + C_2 - \frac{1}{3}(x_1 + x_2 + x_3)^2 - \frac{1}{3}(y_1 + y_2 + y_3)^2 .$$

Hence S will achieve its minimum value,

$$C_1 + C_2 - \frac{1}{3}(x_1 + x_2 + x_3)^2 - \frac{1}{3}(y_1 + y_2 + y_3)^2 \geq 0 ,$$

whenever

$$x = \frac{x_1 + x_2 + x_3}{3} , \quad y = \frac{y_1 + y_2 + y_3}{3} . \tag{3.4}$$

The point (x,y) given in (3.4) is called the centroid or the center of mass of the triangle ABC. We remark that the solution (3.4) is obvious from a physical point of view, for the expression (3.2) is nothing more than the moment of inertia of a system of unit masses at the points A, B, C about a line L which is perpendicular to the plane of A, B, C at the point P, and this is minimized when P is the centroid. Since it is possible to have an intuitive feeling for many of the inequalities with

which we deal, it is worthwhile to define precisely the physical concepts associated with some of the inequalities.

<u>The Moments of a Point-Mass System</u>. Let $P_1, P_2, \cdots, P_n$ be a set of n points in the (x,y)-plane, having coordinates $(x_1,y_1), (x_2,y_2), \cdots, (x_n,y_n)$, and suppose that there is a non-negative mass m_i $(i = 1,2,\cdots,n)$ located at the point P_i. Let L be a line in the (x,y)-plane, and let d_i be the (signed) distance from P_i to L. The sum

$$m_1d_1 + m_2d_2 + \cdots + m_nd_n \tag{3.5}$$

is defined to be the <u>total first moment</u> of the point-mass system with respect to L. More generally, for a positive integer k, the sum

$$m_1d_1^k + m_2d_2^k + \cdots + m_nd_n^k \tag{3.6}$$

is defined to be the <u>total k-th moment</u> of the system with respect to L. For the case k = 2, the expression (3.6) is called the <u>moment of inertia</u> of the system with respect to L.

Let us take two perpendicular lines in the plane, the x-axis and the y-axis, and consider the total first moments of the system with respect to these two lines:

$$\begin{cases} M_1 = m_1x_1 + m_2x_2 + \cdots + m_nx_n \\ M_2 = m_1y_1 + m_2y_2 + \cdots + m_ny_n \, . \end{cases} \tag{3.7}$$

We now ask the question:

(3.8) <u>Is there a point</u> $P(\overline{x},\overline{y})$ <u>such that, if the entire mass of the system were located at</u> P, <u>the total first moments</u> M_1 <u>and</u> M_2 <u>are the same?</u>

There clearly exists such a point, for, since the total mass of the system is $m_1 + m_2 + \cdots + m_n$, and

$$(m_1 + m_2 + \cdots + m_n)\overline{x} = M_1$$
$$(m_1 + m_2 + \cdots + m_n)\overline{y} = M_2 ,$$

so that, by (3.7),

$$(3.9)\quad \begin{cases} \overline{x} = \dfrac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n} = \dfrac{\sum_{i=1}^{n} m_ix_i}{\sum_{i=1}^{n} m_i} \\[2em] \overline{y} = \dfrac{m_1y_1 + m_2y_2 + \cdots + m_ny_n}{m_1 + m_2 + \cdots + m_n} = \dfrac{\sum_{i=1}^{n} m_iy_i}{\sum_{i=1}^{n} m_i} . \end{cases}$$

The point P with coordinates given by (3.9) is called the center of gravity or center of mass of the system. If L is any line passing through the point $P(\overline{x},\overline{y})$, it is not difficult to show (see Problem 3.8) that the total first moment (3.5) of the system with respect to the line L is zero.

Example 5. Given a point-mass system consisting of n masses m_i located at the points P_i, $i = 1, 2, \cdots, n$, and given an arbitrary but fixed line L in the plane. Among all lines parallel to L find that line with respect to which the total second moment--that is, the moment of inertia--is a minimum.

We proceed in the following way: We let L_o be the line through the center of gravity of the system and express the moment of inertia of the system about any other line $\mathcal{L}$ parallel to the given line L in terms of the moment of inertia about L_o. Let

α be the (signed) distance between L_o and $\mathcal{L}$; if the moment of inertia of the system about L_o is

$$(3.10) \qquad I_{L_o} = m_1 d_1^2 + m_2 d_2^2 + \cdots + m_n d_n^2 ,$$

then the moment of inertia of the system about $\mathcal{L}$ is

$$(3.11) \qquad I_{\mathcal{L}} = m_1(d_1 + \alpha)^2 + m_2(d_2 + \alpha)^2 + \cdots + m_n(d_n + \alpha)^2 .$$

If we expand the squares in (3.11), we have

$$I_{\mathcal{L}} = I_{L_o} + 2\alpha \sum_{i=1}^{n} m_i d_i + \alpha^2 \sum_{i=1}^{n} m_i ,$$

where the first summation is the total first moment with respect to L_o and the second summation is the total mass M of the system. Since L_o passes through the center of gravity, $\Sigma\, m_i d_i = 0$, and we have the relation

$$(3.12) \qquad I_{\mathcal{L}} = I_{L_o} + \alpha^2 M .$$

Thus the line parallel to L and passing through the center of gravity of the system will minimize the total second moment or moment of inertia.

Exercises

3.1. Find the minimum value of $\dfrac{(3x + 2)(x + 1)}{2x + 1}$ for $x > -\dfrac{1}{2}$.

3.2. Find the maximum value of $\dfrac{7}{x^2 + 4x + 5}$.

3.3. Find the maximum value of $\dfrac{x + 3}{x^2 + 4x + 5}$.

3.4. Find the maximum value of $\dfrac{3}{5 + \cos^4 x + \sin^4 x}$.

3.5. For positive x and y, find the maximum value of

$\dfrac{7}{x^2 + y^2 + 7 + \dfrac{1}{x^2} + \dfrac{1}{y^2}}$ if x and y are constrained by $x + y = 1$.

3.6. For positive x and y, find the maximum value of

$$\frac{18}{\left\{2 + (x + y)\left(\frac{1}{x} + \frac{1}{y}\right)\right\}\left[1 + \sqrt{x + y} + \frac{1}{\sqrt{x + y}}\right]}.$$

3.7. **If A,B,C are the angles of an acute triangle, find the smallest value of the expression $\tan^2 A + \tan^2 B + \tan^2 C$. (Hint: Using the fact that $A + B + C = \pi$, prove first the identity $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.)**

3.8. Prove the assertion following Equations (3.9), namely, that if $P(\bar{x},\bar{y})$ is the center of gravity of a point-mass system, then the total first moment of the system about any line $\mathcal{L}$ passing through P is zero.

3.9. Given a tetrahedron ABCD in 3-dimensional space. Find the point P such that the sum of the squares from P to the vertices A, B, C, D is a minimum.

3.10. Given a 3-dimensional point-mass system consisting of n masses m_i located at points $P_i(x_i, y_i, z_i)$, $i = 1, 2, \cdots, n$. Given a plane Π: $ax + by + cz + d = 0$, we define the total first moment of the system with respect to Π to be the sum

$$d_1 m_1 + d_2 m_2 + \cdots + d_n m_n ,$$

where d_i is the signed distance from P_i to Π (for example, d_i will be taken positive if P_i lies above Π and negative if P_i lies below Π). The question (3.8) is easily formulated in three dimensions, and, with respect to the three coordinate planes, we are led to the coordinates of the center of gravity of the system

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i m_i}{M}, \quad \bar{y} = \frac{\sum_{i=1}^{n} y_i m_i}{M}, \quad \bar{z} = \frac{\sum_{i=1}^{n} z_i m_i}{M},$$

where $M = \sum_{i=1}^{n} m_i$ is the total mass of the system.

Show that the total first moment of the system with respect to any plane passing through the center of gravity is zero.

3.11. For the point-mass system described in Problem (3.10), the quantity

$$m_1 d_1^2 + m_2 d_2^2 + \cdots + m_n d_n^2 ,$$

where d_i is the distance to a given line $\mathcal{L}$ in space, is called the <u>moment of inertia</u> $I_{\mathcal{L}}$ of the system about the line $\mathcal{L}$. Given a regular tetrahedron ABCD and unit masses at the vertices A, B, C, D (that is, $m_i = 1$, $i = 1, 2, 3, 4$); find $I_{\mathcal{L}}$ if $\mathcal{L}$ is the line passing through A and the centroid of the triangle BCD.

3.2. <u>The Inequality of Cauchy and Schwarz</u>. We wish to show another method of using the quadratic function (3.1). Suppose that we are given two sets of real numbers $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$; in this section we lift the condition that the numbers a_k, b_k be positive. For any value of the real number x, the expression

$$y = (a_1x - b_1)^2 + (a_2x - b_2)^2 + \cdots + (a_nx - b_n)^2$$

cannot be negative, i.e.

$$y = \sum_{k=1}^{n} (a_kx - b_k)^2 \geq 0 \; . \tag{3.13}$$

We observe first that equality in (3.13) can hold only if each of the terms $(a_1x - b_1)^2, \cdots, (a_nx - b_n)^2$ is zero, that is, only if

$$\frac{b_1}{a_1} = \frac{b_2}{a_2} = \cdots = \frac{b_n}{a_n} \; . \tag{3.14}$$

We may write (3.13) as

$$y = \left(\sum_{k=1}^{n} a_k^2\right)x^2 - 2\left(\sum_{k=1}^{n} a_kb_k\right)x + \left(\sum_{k=1}^{n} b_k^2\right) \geq 0$$

which is of the form (3.1) with

$$\left\{\begin{aligned} a &= \sum_{k=1}^{n} a_k^2 \\ b &= -2\sum_{k=1}^{n} a_kb_k \\ c &= \sum_{k=1}^{n} b_k^2 \; . \end{aligned}\right. \tag{3.15}$$

Notice that we have imposed on (3.1) the additional condition (3.13), namely, that y cannot be negative. The value of a in (3.15) indicates that the parabola (3.1) must open upwards, and the condition that y cannot be negative means that the parabola cannot cross the x-axis at two distinct points. Consequently, the discriminant $b^2 - 4ac$ of the quadratic

(3.1) cannot be positive, which means that

$$4\left(\sum_{k=1}^{n} a_k b_k\right)^2 - 4\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) \leq 0 ,$$

or

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{b} b_k^2\right) . \tag{3.16}$$

This is the <u>Cauchy-Schwarz inequality</u>, and the condition under which equality holds is given by (3.14). Another form of (3.16) which is useful may be obtained by taking the square roots of the terms in (3.16):

$$\left|\sum_{k=1}^{n} a_k b_k\right| \leq \left(\sum_{k=1}^{n} a_k^2\right)^{1/2}\left(\sum_{k=1}^{n} b_k^2\right)^{1/2} . \tag{3.17}$$

The right-hand side of (3.17) is geometrically suggestive, since each term may be interpreted as a distance when n is 2 or 3. The following example illustrates the geometrical meaning.

<u>Example 6</u>. Given the points $A(a_1,a_2,a_3)$ and $B(b_1,b_2,b_3)$ in a 3-dimensional space. Find the angle between $\overline{OA}$ and $\overline{OB}$, where O denotes the origin (0,0,0). The three sides of triangle OAB may be found by the distance formula:

$$\begin{cases} OA = \sqrt{a_1^2 + a_2^2 + a_3^2} = \left(\sum_{k=1}^{3} a_k^2\right)^{1/2} \\ OB = \sqrt{b_1^2 + b_2^2 + b_3^2} = \left(\sum_{k=1}^{3} b_k^2\right)^{1/2} \\ AB = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2} = \left(\sum_{k=1}^{3} (a_k-b_k)^2\right)^{1/2} , \end{cases} \tag{3.18}$$

and we may find the angle α between OA and OB by means of the law of cosines,

$$AB^2 = OA^2 + OB^2 - 2OA \cdot OB \cos \alpha,$$

by substituting the expressions in (3.18):

$$(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 = a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 -$$

$$- 2OA \cdot OB \cos \alpha.$$

If we expand the terms on the left-hand side and make the obvious cancellations, we are left with

$$OA \cdot OB \cos \alpha = a_1b_1 + a_2b_2 + a_3b_3 ,$$

or

$$\cos \alpha = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}} . \tag{3.19}$$

The formula (3.19) for the angle α shows, at least in dimensions 2 and 3, the meaning of the Cauchy-Schwarz inequality (3.17): the cosine of an angle lies between -1 and +1, and can be 1 only when the angle is zero. The fact that we have a purely arithmetical proof of (3.17) allows us to introduce the notion of an angle into higher-dimensional spaces; indeed, this is precisely how one may set up an analytic geometry in higher-dimensional spaces.

Example 7. Use the Cauchy-Schwarz inequality to find the minimum value of $A \csc^2 x + B \sec^2 x$ if $A > 0$, $B > 0$, $0 < x < \frac{\pi}{2}$. We have

$$A \csc^2 x + B \sec^2 x = (A \csc^2 x + B \sec^2 x)(\sin^2 x + \cos^2 x)$$

$$\geq (\sqrt{A} \csc x \sin x + \sqrt{B} \sec x \cos x)^2,$$

or

$$A \csc^2 x + B \sec^2 x \geq (\sqrt{A} + \sqrt{B})^2 ,$$

with equality if and only if

$$\frac{\sqrt{A} \csc x}{\sin x} = \frac{\sqrt{B} \sec x}{\cos x} ,$$

or $\tan^2 x = \sqrt{\frac{A}{B}}$, or $x = \arctan \sqrt[4]{\frac{A}{B}}$.

Example 8. Find the minimum value of the function $(16 + x^2 + y^4)(\frac{1}{16} + \frac{1}{x^2} + \frac{1}{y^4})$. This expression is essentially the right-hand side of (3.16) with $a_1 = 4$, $a_2 = x$, $a_3 = y^2$ and $b_1 = \frac{1}{4}$, $b_2 = \frac{1}{x}$, $b_3 = \frac{1}{y^2}$. Hence

$$(16 + x^2 + y^4)(\frac{1}{16} + \frac{1}{x^2} + \frac{1}{y^4}) \geq (1 + 1 + 1)^2 = 9,$$

with equality if

$$\frac{4}{1/4} = \frac{x}{1/x} = \frac{y^2}{1/y^2} \quad \text{or} \quad x = 4, \quad y = 2.$$

In examples of this type, there is an alternative solution involving the inequality between arithmetic and geometric means. For the first factor we have

$$16 + x^2 + y^4 \geq 3 \sqrt[3]{16x^2y^4} ,$$

while, for the second, we have

$$\frac{1}{16} + \frac{1}{x^2} + \frac{1}{y^4} \geq 3\sqrt[3]{\frac{1}{16x^2y^4}} .$$

If we multiply these two inequalities, we obtain

$$(16 + x^2 + y^4)\left(\frac{1}{16} + \frac{1}{x^2} + \frac{1}{y^4}\right) \geq 9\sqrt[3]{16x^2y^4} \cdot \sqrt[3]{\frac{1}{16x^2y^4}} = 9,$$

with equality only if $16 = x^2 = y^4$; this result coincides with our first solution.

The inequality of Cauchy and Schwarz may be generalized in many ways, and several of these extensions are interchangeable in their application to elementary geometry. We wish to introduce here a simple form of a classical inequality which will be treated at length in Chapter 4; this is a special case of Minkowski's inequality. Let $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ be two triples of real numbers; then

$$\left(\sum_{k=1}^{3} (a_k + b_k)^2\right)^{1/2} \leq \left(\sum_{k=1}^{3} a_k^2\right)^{1/2} + \left(\sum_{k=1}^{3} b_k^2\right)^{1/2}, \tag{3.20}$$

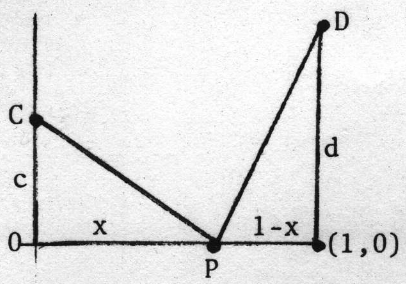
with equality occurring only if

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} . \tag{3.21}$$

The proofs of (3.20) and (3.21) are obvious on the basis of our discussions in Example 6 above: If we form the parallelogram of which two sides are OA and OB, then the fourth vertex will be the point $C(a_1 + b_1, a_2 + b_2, a_3 + b_3)$, and the length of OC will be

$\sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2 + (a_3 + b_3)^2}$. In the triangle OAC, we shall have $OA + AC \geq OC$ with equality only if the point A lies on OC. Since $AC = OB$, (3.20) and (3.21) follow at once. In other words, (3.20) asserts that the sum of two sides of a triangle must exceed the third side.

Example 9. From the point $C(0,c)$ a man runs to a point $P(x,0)$, $0 \leq x \leq 1$, and then to the point $D(1,d)$ (see the figure). Find the point P so that $CP + PD$ is a minimum.



The distance to be covered is

$$CP + PD = \sqrt{c^2 + x^2} + \sqrt{d^2 + (1 - x)^2},$$

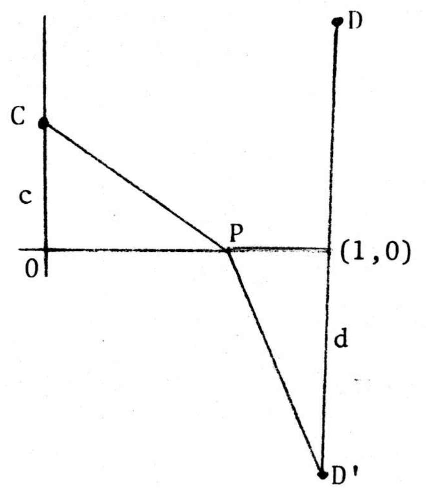
and by (3.20), the sum of the two radicals is greater than or equal to

$$\sqrt{(c + d)^2 + (x + 1 - x)^2} = \sqrt{(c + d)^2 + 1}\ .$$

Thus $CP + PD$ will assume its least value $\sqrt{(c + d)^2 + 1}$ when the equality (3.21) holds, or when

$$\frac{c}{d} = \frac{x}{1 - x} \quad \text{or} \quad x = \frac{c}{c + d}, \quad 1 - x = \frac{d}{c + d}\ .$$

There is a particularly elegant geometrical solution to this problem, which brings out the fact that Minkowski's inequality, in the form (3.20), says essentially that the shortest distance from one point to another is the straight-line segment between the two points. Reflect the point D



in the x-axis into the point D' (1,-d). Then the choice of the point P(x,0) to minimize the sum CP + PD will also minimize the sum CP + PD'. To minimize the latter sum, P must lie on the straight-line segment drawn from C to D'. By elementary geometry, it then follows that the minimum is $\sqrt{(c + d)^2 + 1}$ and that this minimum occurs when $x = \frac{c}{c + d}$ and $1 - x = \frac{d}{c + d}$.

The additive property of the summation sign in (3.13) suggests that we may exploit even further the quadratic function (3.1). We shall assume some knowledge of the elementary integral calculus, even though, with the exception of the Lemma below, the following developments are conceptually identical to the derivation of (3.13), (3.15), and (3.16) above.

Suppose that we have two functions f(x) and g(x) which are both continuous on the closed interval $a \leq x \leq b$. For any real number t, the expression $[f(x)t - g(x)]^2$ cannot be negative, and is zero for all x in $a \leq x \leq b$ only if $g(x) = f(x)t$, that is, only if g(x) is a real multiple of f(x). Consequently, we have

$$(3.22) \qquad y = \int_a^b [f(x)t - g(x)]^2 \, dx \geq 0$$

for all real values of t; this is the analogue of (3.13). The conditions under which equality prevails in (3.22) are not arrived at as simply as in (3.13); and we reserve comment until we have obtained the Lemma below.

The inequality (3.22) may be written

$$y = \int_a^b ([f(x)]^2 t^2 - 2t\, f(x)\, g(x) + [g(x)]^2) dx \geq 0$$

or

(3.23) $$y = \left(\int_a^b [f(x)]^2\, dx\right) t^2 - 2\left(\int_a^b f(x)g(x)dx\right) t + \int_a^b [g(x)]^2\, dx \geq 0.$$

Here we have again a non-negative quadratic function $y = a't^2 + b't + c'$ of the form (3.1), where we have used t in place of x, since x is reserved as the "dummy" variable in the integral; the coefficients of the quadratic are

(3.24) $$\begin{cases} a' = \int_a^b [f(x)]^2\, dx \\ b' = -2\int_a^b f(x)g(x)dx \\ c' = \int_a^b [g(x)]^2\, dx\,. \end{cases}$$

Since the quadratic function (3.23) can never be negative, it cannot have distinct real roots, so that the discriminant $b'^2 - 4a'c'$ cannot be positive:

$$4\left(\int_a^b f(x)g(x)dx\right)^2 - 4\left(\int_a^b [f(x)]^2\, dx\right)\left(\int_a^b [g(x)]^2\, dx\right) \leq 0\,,$$

or

(3.25) $$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \left(\int_a^b [f(x)]^2\, dx\right)\left(\int_a^b [g(x)]^2\, dx\right),$$

which is called Schwarz's inequality.

We now take up the question of when equality can occur in (3.25), and this is equivalent to equality in (3.22). The following Lemma is important in many contexts, and we emphasize the fact that the condition of continuity is essential.

LEMMA. Let $F(x)$ be continuous and non-negative on the interval $a \leq x \leq b$. If $\int_a^b F(x)dx = 0$, then $F(x)$ is identically zero.

Proof. Suppose, to the contrary, that $F(x)$ is not identically zero. Then there is a point x_o such that $F(x_o) \neq 0$. Let us denote the value $F(x_o)$ by A; we must have that $A > 0$ since $F(x)$ is non-negative. Because $F(x)$ is continuous, there exists an interval $|x - x_o| < \delta$, or $x_o - \delta < x < x_o + \delta$, such that $F(x) > A/2$ for every x in this interval. (If x_o should be one of the end-points, a or b, it will be enough to take half of the stipulated interval, e.g., $b - \delta < x < b$ if $x_o = b$; the ensuing argument will also apply to this interval.) Since $F(x)$ is non-negative, we have

$$0 = \int_a^b F(x)dx \geq \int_{x_o-\delta}^{x_o+\delta} F(x)dx > \int_{x_o-\delta}^{x_o+\delta} \frac{A}{2}\, dx$$

$$= \frac{A}{2} \int_{x_o-\delta}^{x_o+\delta} dx = \frac{A}{2}\,(2\delta) = A\delta > 0 \,,$$

and from the contradiction (namely, that $0 > 0$), we have that $F(x)$ is identically zero. (If x_o had been one of a and b, then our last step in the chain of inequalities above would have yielded $A\delta/2$, rather than $A\delta$.)

To complete the discussion of Schwarz's inequality, we saw that the non-negative quadratic function arising from (3.22) could not have two distinct real roots. However, equality may occur in (3.22), and therefore in (3.25), if the quadratic function (3.22) possesses one real root, say t_o. Then for this real root t_o, we have $\int_a^b [f(x)t_o - g(x)]^2\,dx = 0$. The function $[f(x)t_o - g(x)]^2$ is continuous and may therefore play the role of $F(x)$ in the Lemma. Hence $[f(x)t_o - g(x)]^2$ is identically zero, so that, for all x in $a \le x \le b$, $g(x) = t_o f(x)$.

Example 10. Let us consider the integral $\int_o^1 x^4\,dx$, which has the value 1/5. We may write x^4 as $x \cdot x^3$, and if we take $f(x) = x$ and $g(x) = x^3$, we have

$$\frac{1}{25} = \left(\int_o^1 x^4 dx\right)^2 = \left(\int_o^1 x \cdot x^3 dx\right)^2 < \left(\int_o^1 x^2 dx\right)\left(\int_o^1 x^6 dx\right) = \frac{1}{3} \cdot \frac{1}{7} = \frac{1}{21} .$$

On the other hand, we may write $x^4 = x^2 \cdot x^2$ and take $f(x) = x^2$ and $g(x) = x^2$. In this case, we have

$$\frac{1}{25} = \left(\int_o^1 x^4 dx\right)^2 = \left(\int_o^1 x^2 \cdot x^2 dx\right)^2 \le \left(\int_o^1 x^4 dx\right)\left(\int_o^1 x^4 dx\right) = \frac{1}{5} \cdot \frac{1}{5} = \frac{1}{25} .$$

In this second instance, it is obvious that the equality occurs because $f(x)$ is a constant multiple of $g(x)$.

Example 11. Show that, for $x > 1$, $\log x > \dfrac{2(x - 1)}{x + 1}$. (This is a somewhat contrived illustration, since it is not at all obvious how the inequality is related to Schwarz's inequality, but it illustrates, after the fact, how the inequality may be applied.) We have

$$(x - 1)^2 = \left(\int_1^x dt\right)^2 = \left(\int_1^x \frac{1}{\sqrt{t}}\sqrt{t}\, dt\right)^2 < \left(\int_1^x \frac{1}{t}\, dt\right)\left(\int_1^x t\, dt\right)$$

$$= (\log x)\left(\frac{1}{2}(x^2 - 1)\right), \quad \text{whence}$$

$$\frac{2(x - 1)^2}{x^2 - 1} < \log x, \quad \text{or} \quad \frac{2(x - 1)}{x + 1} < \log x.$$

Set $x = e$, and we have that $\frac{2e - 2}{e + 1} < 1$, or $e < 3$. If we set $x = e^{1/3}$ in this inequality, we obtain

$$\frac{2(e^{1/3} - 1)}{e^{1/3} + 1} < \frac{1}{3}\log e = \frac{1}{3}, \quad \text{or} \quad e < \frac{7^3}{5^3} = 2.744.$$

We remark that this result is useful in the modern calculus course, where log x is defined as $\int_1^x dt/t$, and where the inequality of Schwarz is easily proved at the time that the definite integral is introduced. Of course, the inequality of Example 11 may be obtained from the Taylor-series expansion, but this concept usually appears much later in the introductory calculus course.

An easy lower bound for e may be obtained by analogous methods. From the inequality

$$x^2 = \left(\int_0^x e^{\frac{t}{2}} e^{-\frac{t}{2}}\, dt\right)^2 < \left(\int_0^x e^t\, dt\right)\left(\int_0^x e^{-t}\, dt\right)$$

$$= (e^x - 1)(-e^{-x} + 1) = e^x + \frac{1}{e^x} - 2,$$

we have

$$e^x > \frac{x^2 + 2 + x\sqrt{x^2 + 4}}{2},$$

and, if we set $x = 1$, we have that $e > \frac{3 + \sqrt{5}}{2} \approx 2.62$. If we set $x = 1/3$, for example, we have $e > 2.705$.

Exercises for Chapter III

3.12. Show that $(x + y + z)(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}) \geq 9$ for positive x, y, z.

3.13. Show that $(x_1 + x_2 + \cdots + x_n)(\frac{1}{x_1} + \cdots + \frac{1}{x_n}) \geq n^2$ for positive $x_1, \cdots, x_n$.

3.14. For positive values of A and B find the minimum value of

$$\frac{A}{1 - \cos x} + \frac{B}{1 + \cos x} \qquad (0 < x < \pi).$$

3.15. Find the largest value of $3x + 4\sqrt{4 - x^2}$ in the interval $0 \leq x \leq 2$.

3.16. Find the largest and smallest values of $12x + 3y + 4z$ if the point (x,y,z) is constrained by the relation $x^2 + y^2 + z^2 = 1$.

3.17. Find the minimum value of $3\sec^6 x + 3\csc^6 x$ for $0 < x < \pi/2$.

3.18. Show that $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} \leq \sqrt{\frac{x^2}{2} + \frac{y^2}{3} + \frac{z^2}{6}}$.

3.19. Show that $\frac{x}{y + z} + \frac{y}{z + x} + \frac{z}{x + y} \geq 3/2$ for positive x, y, z.

3.20. Given the points $A(0,0,a)$ and $B(1,1,b)$ with $a > 0$ and $b > 0$. Use the inequality (3.20) to find that point $P(x,y,0)$ in the (x,y)-plane so that $AP + PB$ is a minimum. Verify your result with an elementary geometrical solution.

3.21. Show that $|xy + yz + zx| \leq x^2 + y^2 + z^2$.

3.22. Given four triples of positive numbers, (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3), (d_1, d_2, d_3). Show that

$$(a_1b_1c_1d_1 + a_2b_2c_2d_2 + a_3b_3c_3d_3)^4 \leq$$

$$(a_1^4 + a_2^4 + a_3^4)(b_1^4 + b_2^4 + b_3^4)(c_1^4 + c_2^4 + c_3^4)(d_1^4 + d_2^4 + d_3^4).$$

3.23. For x in the range $0 < x < \pi/2$, show that $\dfrac{4x^2}{2x + \sin 2x} < \tan x$.

(Hint. Start with the identity $x = \int_0^x \sec t \cos t \, dt$.)

3.24. Show that, if $x_1, \cdots, x_n$ are real,

$$(x_1 + x_2 + \cdots + x_n)^2 \leq n(x_1^2 + x_2^2 + \cdots + x_n^2),$$

and compare the result with (2.5).

3.25. Let $a_1, a_2, \cdots, a_n$ be positive numbers such that $\sum_{i=1}^{n} a_i = 1$. Show that if $x_1, x_2, \cdots, x_n$ are real, then

$$(a_1x_1 + a_2x_2 + \cdots + a_nx_n)^2 \leq (a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2).$$

(Compare the result with Problems 3.18 and 3.24.)

3.26. Find the largest and smallest values of $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ for $a_1, a_2, \cdots, a_n$ positive if

$$\frac{x_1^2}{b_1^2} + \frac{x_2^2}{b_2^2} + \cdots + \frac{x_n^2}{b_n^2} = 1.$$

3.27. Let $p(x)$ be continuous and positive on the interval $a \leq x \leq b$, and let $f(x)$ and $g(x)$ be continuous on $a \leq x \leq b$. Prove that

$$\left(\int_a^b f(x)g(x)p(x)dx\right)^2 \leq \left(\int_a^b [f(x)]^2 p(x)dx\right)\left(\int_a^b [g(x)]^2 p(x)dx\right),$$

with equality only if $g(x)$ is a constant multiple of $f(x)$. The positive function $p(x)$ is called a weight function for this form of Schwarz's inequality.

3.28. Given two sets of real numbers $\{a_1, \cdots, a_n\}$ and $\{b_1, \cdots, b_n\}$. Prove the Cauchy-Schwarz inequality from the fact that the double sum

$$\sum_{j=1}^{n} \sum_{k=1}^{n} (a_j b_k - a_k b_j)^2$$

is non-negative.

3.29. Given two infinite sequences of real numbers, $\{a_1, a_2, \cdots, a_n, \cdots\}$ and $\{b_1, b_2, \cdots, b_n, \cdots\}$, such that the series $\sum_{k=1}^{\infty} a_k^2$ and $\sum_{k=1}^{\infty} b_k^2$ are convergent. Show that the infinite series $\sum_{k=1}^{n} a_k b_k$ is absolutely convergent.

3.30. Suppose that $f(x)$ and $f'(x)$ are continuous for $0 \leq x < \infty$ and that the integrals $\int_0^{\infty} [f(x)]^2 dx$ and $\int_0^{\infty} [f'(x)]^2 dx$ are finite with $f(x)$ the integral of $f'(x)$. Show that $\lim_{x \to \infty} f(x) = 0$.

Chapter IV. The Inequalities of Hölder and Minkowski

4.1. A Lemma. Let p be a real number greater than 1, and let q be the real number defined by

$$(4.1)\qquad \frac{1}{p} + \frac{1}{q} = 1 \; ;$$

it follows at once that $q > 1$. Our principal tool in this chapter will be the inequality

$$(4.2)\qquad \frac{x^p}{p} + \frac{y^q}{q} \geq xy \; ,$$

with equality only if $x^p = y^q$, where x and y are any positive real numbers. We note that when $p = q = 2$, (4.1) is satisfied and (4.2) becomes

$$\frac{x^2}{2} + \frac{y^2}{2} \geq xy \; ,$$

an inequality relating the arithmetic and geometric means. We shall show that (4.2) may be deduced directly for rational p and q from the results of Chapter 2 on arithmetic and geometric means. Then we shall show that, for two sets of positive real numbers, $\{a_1, a_2, \cdots, a_n\}$ and $\{b_1, b_2, \cdots, b_n\}$,

$$(4.3)\qquad a_1b_1 + a_2b_2 + \cdots + a_nb_n \leq \left(a_1^p + a_2^p + \cdots + a_n^p\right)^{1/p} \left(b_1^q + b_2^q + \cdots + b_n^q\right)^{1/q} ,$$

with equality if and only if

$$(4.4) \qquad \frac{a_1^p}{b_1^q} = \frac{a_2^p}{b_2^q} = \ldots = \frac{a_n^p}{b_n^q} .$$

The inequality (4.3) is called Hölder's inequality, which we shall generalize in what follows. We wish to remark that Hölder's inequality is an important extension of the inequality of Cauchy and Schwarz.

We wish to emphasize here the fact that when p and q are rational, we are able to deduce (4.2) and (4.3) by so-called "elementary methods", that is, by methods not involving analysis (or infinite processes). Of course, when p is irrational, the functions x^p and y^q in (4.2) cannot be defined without the use of an infinite or limiting process (for example, x^p can be defined as $e^{p(\log x)}$), so that so-called "analytic methods" are necessary. Certain analytic methods will also yield the same results in the case that p and q are rational, but, as we have remarked, it is one of our objects to show that the inequalities of Hölder and Minkowski follow directly from the inequalities relating the arithmetic and geometric means whenever p and q are rational.

To prove (4.2) for positive x and y and for rational $p = m/n > 1$, we consider the expression for the positive integers m and n:

$$\frac{x^p}{p} + \frac{y^q}{q} = \frac{nx^{\frac{m}{n}} + (m - n)y^{\frac{m}{m-n}}}{m} \qquad (\frac{1}{q} = 1 - \frac{1}{p} = \frac{m - n}{m}) ,$$

which, by the inequality between arithmetic and geometric means, is not less than

$$(x^m y^m)^{1/m} = xy ,$$

with equality only for $x^p = y^q$. Hence

$$\frac{x^p}{p} + \frac{y^q}{q} \geq xy \text{ ,}$$

with equality if and only if $x^p = y^q$, which is (4.2).

Now (4.2) holds for real p and q satisfying (4.1). As we mentioned above, when p, and therefore q, is irrational, we must use analytic methods. When one takes the limit or both sides of a strict inequality (<), one must realize that the symbol can degenerate into the "less-than--or-equal" case ($\leq$), e.g. $0 < 1/n$, but $0 \leq \lim\limits_{n\to\infty}(1/n)$. Hence, in the cases involving limits of inequalities, a separate proof of strict inequality is required in those cases where, indeed, strict inequality is possible.

For any irrational p, then, let us consider the expression

$$\frac{x^p}{p} + \frac{y^q}{q} \text{ ,}$$

which was used in the proof of (4.2) for rational p and q, and let us suppose that, for some x and y with $x^p \neq y^q$,

$$\frac{x^p}{p} + \frac{y^q}{q} < xy \text{ ,}$$

and let

$$xy - \left(\frac{x^p}{p} + \frac{y^q}{q}\right) = d > 0 \text{ .} \tag{4.5}$$

We shall show that there exist rational numbers p_o and q_o satisfying (4.1) such that

$$xy - \left(\frac{x^{p_o}}{p_o} + \frac{y^{q_o}}{q_o}\right) \tag{4.6}$$

differs from the expression on the left-hand side of (4.5) by less than d/2; this and the fact that the expression (4.6) with rational exponents is negative by (4.2) then yield a contradiction to (4.5), from which we may conclude that (4.2) holds for arbitrary real p and q. Since x^p and y^q can be approximated arbitrarily closely by rationals, this means that $x^p = x^{p_o} + \varepsilon_1$ and $y^q = y^{q_o} + \varepsilon_2$ for p_o and q_o sufficiently close to p and q, respectively, where ε_1 and ε_2 are arbitrarily small; because of (4.1), ε_2 depends on ε_1, but this relationship offers no difficulty. We have that

$$xy - \left(\frac{x^p}{p} + \frac{y^q}{q}\right) - \left\{xy - \left(\frac{x^{p_o}}{p_o} + \frac{y^{q_o}}{q_o}\right)\right\} =$$

$$= \frac{x^{p_o}}{p_o} - \frac{x^p}{p} + \frac{y^{q_o}}{q_o} - \frac{y^q}{q} = x^{p_o}\left(\frac{1}{p_o} - \frac{1}{p}\right) - \frac{\varepsilon_1}{p} + y^{p_o}\left(\frac{1}{q_o} - \frac{1}{q}\right) - \frac{\varepsilon_2}{q} .$$

Hence the absolute value of the difference on the left-hand side of this last equality does not exceed

$$\frac{x^{p_o}|p - p_o|}{pp_o} + \frac{y^{q_o}\,|q - q_o|}{qq_o} + \frac{|\varepsilon_1|}{p} + \frac{|\varepsilon_2|}{q} < x^{p_o}\,|p - p_o|$$

$$+ y^{q_o}\,|q - q_o| + |\varepsilon_1| + |\varepsilon_2| .$$

Since this last expression tends to zero as p_o tends to p, we can choose p_o close enough to p to ensure that this last expression is less than d/2, which gives us the contradiction that we seek. Hence (4.2) holds for any p and q satisfying (4.1), with equality if and only if $x^p = y^q$.

We remark that the powerful--and unifying--results in Chapter 5 will obviate the need of distinguishing between the rational and irrational cases here and in other contexts.

4.2. <u>Hölder's Inequality</u>. With the establishment of (4.2), we are now in a position to prove Hölder's inequality in the form (4.3). Let $\{a_1, a_2, \cdots, a_n\}$ and $\{b_1, b_2, \cdots, b_n\}$ be two sets of positive numbers, and let p and q be real numbers satisfying (4.1). Let us set

$$(4.7) \qquad \begin{cases} A^p = a_1^p + a_2^p + \cdots + a_n^p \\ B^q = b_1^q + b_2^q + \cdots + b_n^q , \end{cases}$$

so that $(A^p)^{1/p} (B^q)^{1/q} = AB$. We also set

$$a_1 = Ac_1, \quad a_2 = Ac_2, \quad \cdots, \quad a_n = Ac_n$$

$$b_1 = Bd_1, \quad b_2 = Bd_2, \quad \cdots, \quad b_n = Bd_n .$$

From (4.7) we have

$$A^p = A^p c_1^p + A^p c_2^p + \cdots + A^p c_n^p = A^p(c_1^p + c_2^p + \cdots + c_n^p) ,$$

so that

$$(4.8) \qquad c_1^p + c_2^p + \cdots + c_n^p = 1$$

Similarly,

$$(4.9) \qquad d_1^q + d_2^q + \cdots + d_n^q = 1 .$$

By the use of (4.2) we have

$$(4.10)\qquad \begin{cases} a_1b_1 = AB(c_1d_1) \le AB\left(\dfrac{c_1^p}{p} + \dfrac{d_1^q}{q}\right) \\ a_2b_2 = AB(c_2d_2) \le AB\left(\dfrac{c_2^p}{p} + \dfrac{d_2^q}{q}\right) \\ \quad\vdots \\ a_nb_n = AB(c_nd_n) \le AB\left(\dfrac{c_n^p}{p} + \dfrac{d_n^q}{q}\right). \end{cases}$$

If we add the n inequalities of (4.10), we obtain the result

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \le AB\left(\frac{c_1^p + \cdots + c_n^p}{p} + \frac{d_1^q + \cdots + d_n^q}{q}\right),$$

and, by (4.8) and (4.9), we have

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \le AB\left(\frac{1}{p} + \frac{1}{q}\right) = AB,$$

which is precisely Hölder's inequality (4.3). Equality holds in (4.3) if and only if equality holds in each inequality of (4.10), which yields the conditions (4.4).

<u>Remark</u>. The inequality (3.16) of Cauchy and Schwarz given in Chapter 3 is a special case of Hölder's inequality (4.3), where $p = 2$, $q = 2$. Also, since our proof of Hölder's inequality above was based on the inequality between the arithmetic and geometric means, it follows that the Cauchy-Schwarz inequality is also a consequence of the inequality between the arithmetic and geometric means.

4.3. Minkowski's Inequality. In Chapter 3 we encountered the special inequality (3.20),

$$\left(\sum_{k=1}^{3}(a_k \pm b_k)^2\right)^{1/2} \le \left(\sum_{k=1}^{3} a_k^2\right)^{1/2} + \left(\sum_{k=1}^{3} b_k^2\right)^{1/2},$$

which, in the case of triples $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, was easily proved by means of analytic geometry. We wish to prove a generalization of this inequality for two n-tuples of real numbers, $\{a_1, a_2, \cdots, a_n\}$ and $\{b_1, b_2, \cdots, b_n\}$, and for an arbitrary real number $K > 1$. The result is Minkowski's inequality:

$$\left(\sum_{i=1}^{n}(a_i + b_i)^K\right)^{1/K} \le \left(\sum_{i=1}^{n} a_i^K\right)^{1/K} + \left(\sum_{i=1}^{n} b_i^K\right)^{1/K}, \tag{4.11}$$

with equality holding if and only if

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}. \tag{4.12}$$

The proof of (4.11) is a simple application of Hölder's inequality (4.3); we set $p = K$ and $q = K' = \frac{K}{K-1}$, so that $\frac{1}{p} + \frac{1}{q} = 1$. Thus, it follows from Hölder's inequality that

$$\begin{aligned}\sum_{i=1}^{n}(a_i + b_i)^K &= \sum_{i=1}^{n} a_i(a_i + b_i)^{K-1} + \sum_{i=1}^{n} b_i(a_i + b_i)^{K-1}\\ &\le \left(\sum_{i=1}^{n} a_i^K\right)^{1/K}\left(\sum_{i=1}^{n}(a_i + b_i)^{(K-1)K'}\right)^{1/K'}\\ &+ \left(\sum_{i=1}^{n} b_i^K\right)^{1/K}\left(\sum_{i=1}^{n}(a_i + b_i)^{(K-1)K'}\right)^{1/K'}.\end{aligned}$$

If we now use the relationship between K and K' and factor, we have

$$\sum_{i=1}^{n} (a_i + b_i)^K \leq \left(\sum_{i=1}^{n} a_i^K\right)^{1/K} \left(\sum_{i=1}^{n} (a_i + b_i)^K\right)^{\frac{K-1}{K}} +$$

$$+ \left(\sum_{i=1}^{n} b_i^K\right)^{\frac{1}{K}} \left(\sum_{i=1}^{n} (a_i + b_i)^K\right)^{\frac{K-1}{K}}$$

$$= \left\{\left(\sum_{i=1}^{n} a_i^K\right)^{1/K} + \left(\sum_{i=1}^{n} b_i^K\right)^{1/K}\right\} \left(\sum_{i=1}^{n} (a_i + b_i)^K\right)^{1-\frac{1}{K}},$$

or

$$\left(\sum_{i=1}^{n} (a_i + b_i)^K\right)^{1/K} \leq \left(\sum_{i=1}^{n} a_i^K\right)^{1/K} + \left(\sum_{i=1}^{n} b_i^K\right)^{1/K},$$

which is (4.11). The condition for equality, namely (4.12), follows directly from (4.4) if we have simultaneously that

$$\begin{cases} \dfrac{a_1^K}{(a_1 + b_1)^{(K-1)K'}} = \cdots = \dfrac{a_n^K}{(a_n + b_n)^{(K-1)K'}} \\[2ex] \dfrac{b_1^K}{(a_1 + b_1)^{(K-1)K'}} = \cdots = \dfrac{b_n^K}{(a_n + b_n)^{(K-1)K'}} \end{cases},$$

or, since $(K - 1)K' = K$,

$$\frac{a_1}{a_1 + b_1} = \cdots = \frac{a_n}{a_n + b_n} \quad \text{and} \quad \frac{b_1}{a_1 + b_1} = \cdots = \frac{b_n}{a_n + b_n},$$

or, what is the same thing,

$$\frac{a_1}{b_1} = \cdots = \frac{a_n}{b_n}.$$

Minkowski's inequality (4.11) has been proved for arbitrary real $K > 1$. We remark that, whenever K is rational, Minkowski's inequality may be regarded as a direct consequence of the inequality between the arithmetic and geometric means because Hölder's inequality is such a consequence.

4.4. <u>Generalizations of the Inequalities</u>. We begin with the form (4.3) of Hölder's inequality,

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p \right)^{1/p} \left(\sum_{i=1}^{n} b_i^q \right)^{1/q} , \tag{4.13}$$

where p and q are related by the equation

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If we set $1/p = \alpha$ and $1/q = \beta$, and make the substitution $A_i = a_i^p$ and $B_i = b_i^q$, so that $a_i = A_i^\alpha$ and $b_i = B_i^\beta$, we may write (4.13) in the form

$$\sum_{i=1}^{n} A_i^\alpha B_i^\beta \leq \left(\sum_{i=1}^{n} A_i \right)^\alpha \left(\sum_{i=1}^{n} B_i \right)^\beta . \tag{4.14}$$

The condition for equality in (4.14) now takes the form

$$\frac{A_1}{B_1} = \frac{A_2}{B_2} = \cdots = \frac{A_n}{B_n} . \tag{4.15}$$

Suppose now that we are given three n-tuples of positive numbers $\{A_1, \cdots, A_n\}$, $\{B_1, \cdots, B_n\}$ and $\{C_1, \cdots, C_n\}$, together with three positive numbers α, β, γ such that $\alpha + \beta + \gamma = 1$. Let us consider the

expression $\sum_{i=1}^{n} A_i^{\alpha} B_i^{\beta} C_i^{\gamma}$. If we write

$$B_i^{\beta} C_i^{\gamma} = \left(B_i^{\frac{\beta}{1-\alpha}} C_i^{\frac{\gamma}{1-\alpha}} \right)^{1-\alpha} = (D_i)^{1-\alpha} ,$$

we have, by (4.14), that

$$\sum_{i=1}^{n} A_i^{\alpha} D_i^{1-\alpha} \leq \left(\sum_{i=1}^{n} A_i \right)^{\alpha} \left(\sum_{i=1}^{n} D_i \right)^{1-\alpha} ,$$

with equality only if

$$\frac{A_1}{D_1} = \frac{A_2}{D_2} = \cdots = \frac{A_n}{D_n} .$$

Applying (4.14) to $\sum_{i=1}^{n} D_i = \sum_{i=1}^{n} B_i^{\frac{\beta}{1-\alpha}} C_i^{\frac{\gamma}{1-\alpha}}$ after observing $\frac{\beta}{1 - \alpha} + \frac{\gamma}{1 - \alpha} = 1$, we have that

$$\sum_{i=1}^{n} B_i^{\frac{\beta}{1-\alpha}} C_i^{\frac{\gamma}{1-\alpha}} \leq \left(\sum_{i=1}^{n} B_i \right)^{\frac{\beta}{1-\alpha}} \left(\sum_{i=1}^{n} C_i \right)^{\frac{\gamma}{1-\alpha}} ,$$

so that

$$\left(\sum_{i=1}^{n} B_i^{\frac{\beta}{1-\alpha}} C_i^{\frac{\gamma}{1-\alpha}} \right)^{1-\alpha} \leq \left(\sum_{i=1}^{n} B_i \right)^{\beta} \left(\sum_{i=1}^{n} C_i \right)^{\gamma} ,$$

with equality only if

$$\frac{B_1}{C_1} = \frac{B_2}{C_2} = \cdots = \frac{B_n}{C_n} .$$

This implies, finally, that

$$(4.16) \qquad \sum_{i=1}^{n} A_i^{\alpha} B_i^{\beta} C_i^{\gamma} \leq \left(\sum_{i=1}^{n} A_i\right)^{\alpha} \left(\sum_{i=1}^{n} B_i\right)^{\beta} \left(\sum_{i=1}^{n} C_i\right)^{\gamma},$$

with equality only if $\{A_1, \cdots, A_n\}$, $\{B_1, \cdots, B_n\}$, $\{C_1, \cdots, C_n\}$ are proportional sets. This form of Hölder's inequality may be generalized to any number of n-tuples of positive numbers $\{x_{1m}, x_{2m}, \cdots, x_{nm}\}$, $m = 1, 2, \cdots, N$, for any set $\{\alpha_1, \alpha_2, \cdots, \alpha_N\}$ of positive real numbers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_N = 1$, namely,

$$(4.17) \qquad \sum_{i=1}^{n} x_{i1}^{\alpha_1} x_{i2}^{\alpha_2} \cdots x_{iN}^{\alpha_N} \leq \left(\sum_{i=1}^{n} x_{i1}\right)^{\alpha_1} \left(\sum_{i=1}^{n} x_{i2}\right)^{\alpha_2} \cdots \left(\sum_{i=1}^{n} x_{iN}\right)^{\alpha_N}.$$

The proof is by induction, and the method of deriving (4.16) from (4.14) may be used.

We may change (4.16) to a form similar to (4.3) for certain applications. Starting with (4.16), let us set $A_i^{\alpha} = a_i$, $B_i^{\beta} = b_i$ and $C_i^{\gamma} = c_i$, so that, if $1/\alpha = p$, $1/\beta = q$, $1/\gamma = r$, we have $A_i = a_i^p$, $B_i = b_i^q$, $C_i = c_i^r$, with

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1.$$

Then (4.16) becomes

$$(4.18) \qquad \sum_{i=1}^{n} a_i b_i c_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q} \left(\sum_{i=1}^{n} c_i^r\right)^{1/r},$$

with equality only if the n-tuples $\{a_1^p, \cdots, a_n^p\}$, $\{b_1^q, \cdots, b_n^q\}$, and $\{c_1^r, \cdots, c_n^r\}$ are proportional sets. A similar change may be effected with (4.17).

Example 1. Given nine positive numbers with $\dfrac{a_1b_1c_1 + a_2b_2c_2 + a_3b_3c_3}{3} = 1$. Show that

$$\left(\frac{a_1^2 + a_2^2 + a_3^2}{3}\right)^3 \left(\frac{b_1^3 + b_2^3 + b_3^3}{3}\right)^2 \left(\frac{c_1^6 + c_2^6 + c_3^6}{3}\right) \geq 1.$$

From (4.18), we have, since $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$,

$$(a_1^2 + a_2^2 + a_3^2)^{1/2} (b_1^3 + b_2^3 + b_3^3)^{1/3} (c_1^6 + c_2^6 + c_3^6)^{1/6} \geq$$

$$\geq a_1b_1c_1 + a_2b_2c_2 + a_3b_3c_3 = 3,$$

with equality if and only if $a_1^2 : b_1^3 : c_1^6 = a_2^2 : b_2^3 : c_2^6 = a_3^2 : b_3^3 : c_3^6 =$ $= x : y : 1$, say, where $x^{1/2} y^{1/3} (c_1^6 + c_2^6 + c_3^6) = 3$. Raising both sides of this inequality to the sixth power and dividing by 3^6, we arrive at the desired inequality.

We may also generalize Minkowski's inequality even more easily. Let us take inequality (4.11),

$$\left(\sum_{i=1}^{n} (a_i + b_i)^K\right)^{1/K} \leq \left(\sum_{i=1}^{n} a_i^K\right)^{1/K} + \left(\sum_{i=1}^{n} b_i^K\right)^{1/K},$$

with equality if and only if $\dfrac{a_i}{b_i}$ is constant, add $(\sum_{i=1}^{n} c_i^K)^{1/K}$ to both sides and use Minkowski's inequality:

$$\left(\sum_{i=1}^{n} a_i^K\right)^{1/K} + \left(\sum_{i=1}^{n} b_i^K\right)^{1/K} + \left(\sum_{i=1}^{n} c_i^K\right)^{1/K} \geq$$

$$\geq \left(\sum_{i=1}^{n} (a_i + b_i)^K\right)^{1/K} + \left(\sum_{i=1}^{n} c_i^K\right)^{1/K} \geq \left(\sum_{i=1}^{n} (a_i + b_i + c_i)^K\right)^{1/K},$$

with equality in the last inequality if and only if $\frac{a_i + b_i}{c_i} = \frac{b_i}{c_i}\left(\frac{a_i}{b_i} + 1\right)$ is constant. Thus,

$$(4.19) \qquad \left(\sum_{i=1}^{n} a_i^K\right)^{1/K} + \left(\sum_{i=1}^{n} b_i^K\right)^{1/K} + \left(\sum_{i=1}^{n} c_i^K\right)^{1/K} \geq \left(\sum_{i=1}^{n} (a_i + b_i + c_i)^K\right)^{1/K},$$

with equality if and only if $\frac{a_i}{b_i}$ is constant and $\frac{b_i}{c_i}$ is constant, which is to say $a_1 : b_1 : c_1 = a_2 : b_2 : c_2 = \cdots = a_n : b_n : c_n$. This is easily extended by induction to

$$\left(\sum_{i=1}^{n} (x_{i1} + x_{i2} + \cdots + x_{iN})^K\right)^{1/K} \leq$$

$$\leq \left(\sum_{i=1}^{n} x_{i1}^K\right)^{1/K} + \left(\sum_{i=1}^{n} x_{i2}^K\right)^{1/K} + \cdots + \left(\sum_{i=1}^{n} x_{iN}\right)^{1/K},$$

with equality if and only if $x_{11} : x_{12} : \cdots : x_{1N} = \cdots = x_{n1} : x_{n2} : \cdots : x_{nN}$.

4.5. The Integral Inequalities of Hölder and Minkowski. We obtain integral analogues of the Hölder and Minkowski inequalities in much the same manner as we obtained the integral analogue (3.25) of the Cauchy--Schwarz inequality (3.17). We follow rather closely the method of proof of the inequality (4.3) in Section 4.1 above.

Let $f(x)$ and $g(x)$ be continuous and non-negative on some interval $a \leq x \leq b$, and let p and q be positive numbers satisfying (4.1); we assume that neither of $f(x)$ and $g(x)$ is identically zero in order to avoid proving a trivial result. Let us set

$$(4.20)\qquad \begin{cases} \int_a^b [f(x)]^p\,dx = A^p \\ \int_a^b [g(x)]^q\,dx = B^q, \end{cases}$$

where neither of A and B is zero; the functions $[f(x)]^p$ and $[g(x)]^q$ are continuous since $f(x)$ and $g(x)$ are. Let us set

$$(4.21)\qquad \begin{cases} F(x) = \dfrac{f(x)}{A} \\ G(x) = \dfrac{g(x)}{B}\ ; \end{cases}$$

again, $F(x)$ and $G(x)$ are obviously continuous. Since $A^p = \int_a^b [f(x)]^p\,dx = \int_a^b A^p [F(x)]^p\,dx$, it follows that $\int_a^b [F(x)]^p\,dx = 1$. Similarly, $\int_a^b [G(x)]^q\,dx = 1$. Now

$$\int_a^b f(x)g(x)dx = \int_a^b AB\,F(x)G(x)dx = AB\int_a^b F(x)G(x)dx,$$

and, by integrating (4.2) with $F(x)$ in place of x and $G(x)$ in place of y, we have

$$AB\int_a^b F(x)G(x)dx \le AB\int_a^b \left\{\frac{[F(x)]^p}{p} + \frac{[G(x)]^q}{q}\right\}dx = AB\left(\frac{1}{p}+\frac{1}{q}\right) = AB.$$

Hence

$$\int_a^b f(x)g(x)dx \le AB = (A^p)^{1/p}\,(B^q)^{1/q},$$

so that, by the definitions of A and B,

$$(4.22) \qquad \int_b^a f(x)g(x)dx \le \left(\int_a^b [f(x)]^p dx\right)^{1/p} \left(\int_a^b [g(x)]^q\right)^{1/q},$$

with equality only if $[f(x)]^p$ is a multiple of $[g(x)]^q$. We remark that (4.22) reduces immediately to the Cauchy-Schwarz inequality (3.25) if $p = q = 2$.

For three (or more) non-negative continuous functions $f(x)$, $g(x)$, $h(x)$ on $a \le x \le b$, and for three positive numbers α, β, γ such that $\alpha + \beta + \gamma = 1$, it is an easy modification of the proof of (4.16) to show that Hölder's inequality in its integral form may be written as

$$(4.23) \qquad \int_a^b f^\alpha g^\beta h^\gamma \, dx \le \left(\int_a^b f\,dx\right)\left(\int_a^b g\,dx\right)\left(\int_a^b h\,dx\right),$$

and we leave this modification for the exercises (see Problem 4.1). For three positive numbers p, q, r with

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1,$$

we also have the alternative form

$$(4.24) \qquad \int_a^b f\,g\,h\,dx \le \left(\int_a^b f^p\,dx\right)^{1/p}\left(\int_a^b g^q\,dx\right)^{1/q}\left(\int_a^b h^r\,dx\right)^{1/r};$$

again, the proof is left to the exercises (see Problem 4.2).

We conclude this chapter with an integral form of Minkowski's inequality; as in the case of (4.11), the proof depends upon Hölder's inequality. Let $f(x)$ and $g(x)$ be continuous and non-negative on the interval $a \le x \le b$, and let K be a real number greater than 1. We

consider the integral

$$\int_a^b [f(x) + g(x)]^K \, dx \ ,$$

and write it in the form

$$\int_a^b f(x)[f(x) + g(x)]^{K-1} \, dx + \int_a^b g(x)[f(x) + g(x)]^{K-1} \, dx \ ,$$

which, by (4.22), is not greater than

$$\left(\int_a^b [f(x)]^K dx\right)^{1/K} \left(\int_a^b [f(x) + g(x)]^{(K-1)K'} dx\right)^{1/K'} +$$

$$+ \left(\int_a^b [g(x)]^K\right)^{1/K} \left(\int_a^b [f(x) + g(x)]^{(K-1)K'} dx\right)^{1/K'} ,$$

where $K' = \frac{K}{K - 1}$, so that $\frac{1}{K} + \frac{1}{K'} = 1$. If we substitute $\frac{K}{K - 1}$ for K' in this last expression, we have

$$\int_a^b [f(x) + g(x)]^K dx \leq$$

$$\left\{ \left(\int_a^b [f(x)]^K dx\right)^{1/K} + \left(\int_a^b [g(x)]^K dx\right)^{1/K} \right\} \left(\int_a^b [f(x) + g(x)]^K dx\right)^{1 - \frac{1}{K}},$$

so that

$$(4.25) \qquad \left(\int_a^b [f(x) + g(x)]^K dx\right)^{1/K} \leq \left(\int_a^b [f(x)]^K dx\right)^{1/K} + \left(\int_a^b [g(x)]^K dx\right)^{1/K},$$

which is the form of Minkowski's inequality which we want. Equality holds only if $f(x) = \alpha g(x)$. Note also that it is a trivial consequence of (4.25) that

$$(4.26) \quad \left(\int_a^b [f(x) + g(x) + h(x)]^K dx\right)^{1/K}$$

$$\leq \left(\int_a^b [f(x)]^K dx\right)^{1/K} + \left(\int_a^b [g(x)]^K dx\right)^{1/K} + \left(\int_a^b [h(x)] dx\right)^{1/K}$$

for three positive functions $f(x)$, $g(x)$, $h(x)$, with equality if $f(x) = \alpha g(x) = \beta h(x)$, for constants α and β.

Exercises for Chapter IV

4.1. Prove the inequality (4.23).

4.2. Prove the inequality (4.24).

4.3. Use Hölder's inequality to show that, for positive x, y, z,

$$3^6 \leq \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)^5 (x^5 + y^5 + z^5).$$

4.4. Show that, for positive x, y, z,

$$27 \leq \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)^2 (x^2 + y^2 + z^2).$$

4.5. Suppose that, for $x \geq -1$, α is a rational number such that $0 < \alpha < 1$. Show that

$$(1 + x)^\alpha \leq 1 + \alpha x,$$

with equality only for $x = 0$.

4.6. For $x \geq -1$, let α be a rational number greater than 1, $\alpha > 1$. Show that

$$(1 + x)^{\alpha} \geq 1 + \alpha x,$$

with equality for $x = 0$.

4.7. Use the result of Problem 4.6 to show that if $a > 0$, $\alpha > 1$, and $x \geq 0$, the minimum of the function $x^{\alpha} - ax$ is $(1 - \alpha)(\frac{a}{\alpha})^{\frac{\alpha}{\alpha-1}}$ and is assumed when $x = (\frac{a}{\alpha})^{\frac{1}{1-\alpha}}$.

4.8. For $x \geq -1$, let α be a negative rational number. Show that

$$(1 + x)^{\alpha} \geq 1 + \alpha x,$$

with equality only for $x = 0$.

4.9. Extend Problem 4.5 (and therefore Problems 4.6 and 4.8) to the case that α is irrational by the use of a sequence of rational numbers $\{r_n\}$ such that $\lim_{n\to\infty} r_n = \alpha$.

4.10. For real numbers x, a, α such that $x \geq 0$, $a > 0$, $\alpha > 1$ find the minimum value of the expression $x^{\alpha} - ax$.

4.11. Use the identity $\cos^2 x + \sin^2 x = 1$ and Hölder's inequality to show that, for $0 \leq x \leq \pi/2$,

$$\cos^6 x + \sin^6 x \geq 1/4,$$

with equality for $x = \pi/4$.

4.12. Using the ideas of Problem 4.11, show that, for $0 \le x \le \pi/2$,

$$\cos^3 x + \sin^3 x \geq 1/\sqrt{2} ,$$

with equality for $x = \pi/4$.

4.13. For x in the interval $0 \leq x \leq \pi/2$ and for any real number $\alpha > 2$ find the minimum value of $\cos^\alpha x + \sin^\alpha x$.

4.14. Let $\phi(x)$ be continuous and positive on $a \leq x \leq b$, and let $f(x)$ and $g(x)$ be continuous and positive on $a \leq x \leq b$. Prove the following form of Hölder's inequality with $\phi(x)$ as weight function:

$$\int_a^b f(x)g(x)\phi(x)dx \leq \left(\int_a^b [f(x)]^p \; \phi(x)dx\right)^{1/p} \left(\int_a^b [g(x)]^q \; \phi(x)dx\right)^{1/q},$$

where p and q are two positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$.

4.15. For a, b, c, positive, find the maximum value of $ax + b\sqrt[n]{c^n - x^n}$, where $n > 1$.

4.16. Use the result of Problem 4.10 to prove the fundamental inequality (4.2) for real p and q satisfying (4.1).

4.17. Let p, q, r be rational numbers such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, and let x, y, z be positive. Prove that

$$\frac{x^p}{p} + \frac{y^q}{q} + \frac{z^r}{r} \geq xyz ,$$

with equality if $x^p = y^q = z^r$.

4.18. Prove the following analogue of Problem 4.14 for sums: Let c_i $(i = 1, 2, \cdots, n)$ be positive, and let $\{a_1, \cdots, a_n\}$ and $\{b_1, \cdots, b_n\}$ be two sets of positive numbers. If p and q are positive real numbers satisfying (4.1), then

$$\sum_{i=1}^{n} a_i b_i c_i \leq \left(\sum_{i=1}^{n} a_i^p c_i \right)^{1/p} \left(\sum_{i=1}^{n} b_i^q c_i \right)^{1/q},$$

with equality only if

$$\frac{a_1^p}{b_1^q} = \cdots = \frac{a_n^p}{b_n^q}.$$

4.19. Show that $(x + 2y + 4z)(\frac{1}{x} + \frac{2}{y} + \frac{4}{z}) \geq 49$, for x, y, z positive.

4.20. Show that, for $x_1, \cdots, x_n$ positive,

$$(x_1 + 2x_2 + \cdots + 2^{n-1} x_n)\left(\frac{1}{x_1} + \frac{2}{x_2} + \cdots + \frac{2^{n-1}}{x_n}\right) \geq (2^n - 1)^2.$$

4.21. For a, b, c, d positive find the minimum value of the expression

$$S = \frac{a}{c + d} + \frac{b}{d + a} + \frac{c}{a + b} + \frac{d}{b + c}.$$

4.22. If $a^2 + b^2 + c^2 = 1$, show that

$$\frac{a^2}{b + c} + \frac{b^2}{c + a} + \frac{c^2}{a + b} \geq \frac{1}{2(a^3 + b^3 + c^3)}.$$

4.23. If $a^2 + b^2 + c^2 = 1$, show that

$$1 \leq (a^{2-p} + b^{2-p} + c^{2-p})^q (a^{2+q} + b^{2+q} + c^{2+q})^p,$$

where p and q are real numbers satisfying (4.1).

4.24. Let p and q be such that $p > 1$ and $q > 1$. Let $f(x)$ and $g(x)$ be positive continuous functions on the finite interval $a \leq x \leq b$. If $1/p + 1/q \leq 1$, show that there exists a constant k such that

$$\int_a^b f\,g\,dx \leq k\left(\int_a^b f^p dx\right)^{1/p}\left(\int_a^b g^q dx\right)^{1/q}.$$

4.25. Suppose that $f(x)$ and $f'(x)$ are continuous for $0 \leq x < \infty$ and that the integrals $\int_0^\infty |f(x)|^p dx$ and $\int_0^\infty |f'(x)|^q dx$ are finite, where p and q are positive numbers with $1/p + 1/q = 1$. If $f(x)$ is the integral of its derivative $f'(x)$, show that $\lim_{x\to\infty} f(x) = 0$. (See Problem 3.30, which is the case for $p = q = 2$.)

4.26. Let $x_1, x_2, \cdots, x_n$ be positive numbers. Show that

$$4(x_1 + 2x_2 + 3x_3 + \cdots + nx_n)\left(\frac{1}{x_1} + \frac{2}{x_2} + \cdots + \frac{n}{x_n}\right) \geq n^2(n + 1)^2.$$

4.27. Let $x_1, \cdots, x_n$ be positive numbers. Show that

$$64\left(\sum_{k=1}^n k^3\, x_k^2\right)\left(\sum_{k=1}^n \frac{k^3}{x_k}\right)^2 \geq n^6(n + 1)^6.$$

Chapter V. Convexity and Jensen's Inequality

5.1. Convex Functions. Let $y = f(x)$ be a function defined in some interval $I: a \leq x \leq b$, and suppose that $f(x)$ has the property that, for any two points x_1 and x_2 in I,

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2} . \tag{5.1}$$

Then we shall say that $f(x)$ is convex on I. We shall assume throughout that $f(x)$ is continuous on I, and we remark only that there are functions of a rather pathological nature which are not continuous on I and which satisfy (5.1). What (5.1) says in intuitive and geometrical terms is that, between x_1 and x_2, the graph of $f(x)$ cannot lie above the chord joining the points $P_1(x_1, f(x_1))$ and $P_2(x_2, f(x_2))$; see Figure 5.1, where m represents the midpoint $\frac{1}{2}(x_1 + x_2)$ of x_1 and x_2. If the inequality in (5.1) is reversed, we shall call the function $f(x)$ concave.

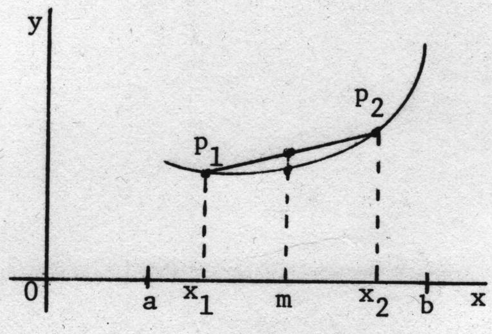


Fig. 5.1

Example 1. Consider the function $f(x) = x^2$ on the interval $I: -\infty < x < \infty$. We have

$$f\left(\frac{x_1 + x_2}{2}\right) = \left(\frac{x_1 + x_2}{2}\right)^2 ; \qquad \frac{f(x_1) + f(x_2)}{2} = \frac{x_1^2 + x_2^2}{2} ,$$

and for all x_1 and x_2 in I, it is an elementary consequence of (1.3) in Chapter 1 that

$$\left(\frac{x_1 + x_2}{2}\right)^2 \leq \frac{x_1^2 + x_2^2}{2},$$

so that (5.1) is fulfilled, and the function x^2 is convex on I.

<u>Example 2</u>. The function $f(x) = x$ is convex on the entire x-axis because (5.1) is satisfied for every x. However, because equality prevails in (5.1), the function x is also concave on the entire x-axis.

It will not be necessary to discuss concave functions separately because of the simple observation that, <u>if</u> f(x) <u>is concave</u>, <u>then</u> -f(x) <u>is convex</u>. It is possible for a simple function to be convex on certain intervals of the x-axis and concave on other intervals; before proceeding to more general considerations, we should like to give an example of the behaviour just described.

<u>Example 3</u>. The function $f(x) = \sin x$ has the property that $-\sin x$ is convex on the interval I: $0 \leq x \leq \pi$. Indeed, from the identity

$$\frac{\sin x_1 + \sin x_2}{2} = \sin \frac{x_1 + x_2}{2} \cos \frac{x_1 - x_2}{2}$$

and from the observation that $\cos \frac{x_1 - x_2}{2} \leq 1$ and $0 \leq \sin \frac{x_1 + x_2}{2}$ on I, it follows that

$$\frac{\sin x_1 + \sin x_2}{2} \leq \sin \frac{x_1 + x_2}{2},$$

or
$$-\sin \frac{x_1 + x_2}{2} \leq -\left(\frac{\sin x_1 + \sin x_2}{2}\right).$$

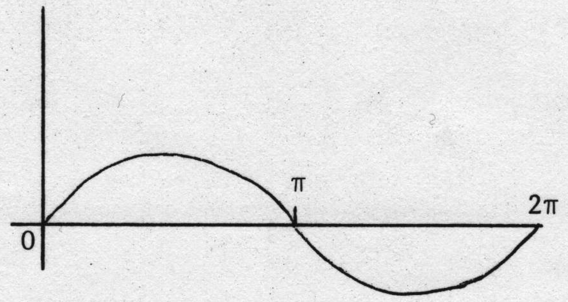


Fig. 5.2

Hence the function $-\sin x$ satisfies (5.1) and is convex on I; equivalently the function $\sin x$ is concave on I. However, on the interval $\pi < x < 2\pi$, the function $\sin x$ is convex (or $-\sin x$ is concave); see Figure 5.2.

It is clear that if we were to devise a method of verifying (5.1) for every given function, we should be confronted with a variety of formidable tasks, each tailored to a specific function and depending on the special properties of that function to determine the convexity (or concavity) of the given function. Consequently, we should like to establish a simple sufficient condition that (5.1) be satisfied. The simplicity and the usefulness of our first condition, which involves elementary notions in the calculus, lead us to a plan of development for this chapter which is somewhat different from the plans of the earlier chapters. In the earlier chapters we developed the inequalities independently of the calculus, although we showed how those inequalities could be applied to integrals as well as to sums. Here we shall give first a simple criterion that (5.1) be satisfied (Theorem 5.1), and then proceed with a development of Jensen's inequality based on (5.1).

THEOREM 5.1. Let $f(x)$ be defined on the interval I: $a \leq x \leq b$ and have a non-negative second derivative $f''(x)$ everywhere on $a < x < b$. Then $f(x)$ is convex on I.

Proof. For any two points x_1 and x_2 in I, the point $\frac{1}{2}(x_1 + x_2)$ is in I. By Taylor's theorem, we have

$$f(x_1) = f\left(\frac{x_1 + x_2}{2}\right) + \left(x_1 - \frac{x_1 + x_2}{2}\right)f'\left(\frac{x_1 + x_2}{2}\right) + \frac{1}{2}\left(x_1 - \frac{x_1 + x_2}{2}\right)^2 f''(\bar{x}_1),$$

where $x_1 < \bar{x}_1 < \frac{1}{2}(x_1 + x_2)$, and

$$f(x_2) = f\left(\frac{x_1 + x_2}{2}\right) + \left(x_2 - \frac{x_1 + x_2}{2}\right)f'\left(\frac{x_1 + x_2}{2}\right) + \frac{1}{2}\left(x_2 - \frac{x_1 + x_2}{2}\right)^2 f''(\bar{x}_2),$$

where $\frac{1}{2}(x_1 + x_2) < \bar{x}_2 < x_2$. If we now add these expressions for $f(x_1)$ and $f(x_2)$ and divide by 2, we have

$$\frac{f(x_1) + f(x_2)}{2} = f\left(\frac{x_1 + x_2}{2}\right) + \frac{1}{16}(x_2 - x_1)^2[f''(\bar{x}_1) + f''(\bar{x}_2)].$$

Now the second term on the right-hand side of this last equation cannot be negative since $f''(x) \geq 0$ by hypothesis, so that

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2},$$

which is (5.1), so that the theorem is proved.

We note, as a corollary to Theorem 5.1, that if $f''(x)$ is never positive on I, then $-f(x)$ is convex, or, equivalently, $f(x)$ is concave on I.

Of all the criteria giving sufficient conditions that a function be convex on an interval I, Theorem 5.1 will turn out to be the simplest and most useful because most of the functions that we shall consider will possess a second derivative at least in certain intervals in which we shall be interested. However, we give some examples to show the type of exceptional case satisfying (5.1) but not possessing even a first derivative.

<u>Example 4</u>. The function $f(x) = |x|$ illustrated in Figure 5.3 fails to have a derivative at $x = 0$, yet has the property that any chord joining two points (x_1,y_1) and $x_2,y_2)$ on the graph cannot lie below the graph, so that the function satisfies (5.1). Let us note another intuitive feature of convexity which will prove useful in the sequel: For each x

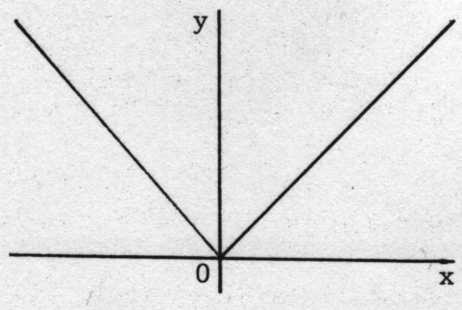


Fig. 5.3

in I, the graph of a convex function on I possesses a <u>line of support</u>, that is, a straight line which intersects the graph of the function in one or more points in such a way that no point of the graph lies below the line of support. In the event that the graph has a tangent at the point x, the tangent line will be the line of support, as in the case of $|x|$ for $x \neq 0$ in Figure 5.3. At the point $x = 0$, any straight line $y = mx$ through the origin will be a line of support provided that m is chosen so that $-1 \leq m \leq 1$.

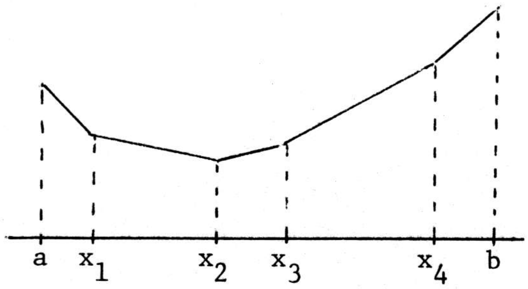


Fig. 5.4

<u>Example 5</u>. Let the graph of f(x) be the polygonal path shown in Figure 5.4, where f(x) is a piecewise linear function. It is clear that (5.1) is satisfied and that f(x) possesses a line of support at every point x in I: $a \leq x \leq b$.

5.2. <u>Jensen's Inequality</u>. The ideas behind Jensen's inequality are quite simple, yet surprisingly far-reaching, and are based solely on the notion that, for a continuous function f(x), the chord between any two points on the graph of f(x) cannot lie below the graph of f(x). This last property of a convex function, which can be expressed analytically as

$$(5.2) \qquad f(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha f(x_1) + (1 - \alpha)f(x_2)$$

for any pair x_1, x_2 on I and for any real number α with $0 \leq \alpha \leq 1$, must now be shown to follow from (5.1) whenever f(x) is continuous.

We begin by showing that, if f(x) satisfies (5.1) in an interval I, then, for any n points $x_1\ x_2, \cdots, x_n$ in I,

$$(5.3) \qquad f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \le \frac{1}{n}[f(x_1) + f(x_2) + \cdots + f(x_n)].$$

The inequality (5.3) is a useful form of Jensen's inequality, and follows directly from (5.1) if n is a power of 2, say $n = 2^k$. For example, if $n = 4 = 2^2$, we have

$$f\left(\frac{x_1 + x_2 + x_3 + x_4}{4}\right) = f\left(\frac{\frac{x_1 + x_2}{2} + \frac{x_3 + x_4}{2}}{2}\right) \le \frac{1}{2}\left\{f\left(\frac{x_1 + x_2}{2}\right) + f\left(\frac{x_3 + x_4}{2}\right)\right\}$$

$$\le \frac{1}{2^2}[f(x_1) + f(x_2) + f(x_3) + f(x_4)].$$

We now proceed, by a form of the induction principle*, to show that (5.3) holds for any positive integer n. Suppose now that n is an arbitrary positive integer; let m be a positive integer such that n + m is a power of 2, say, $n + m = 2^j$. Then, for any point x_o in I,

$$f\left(\frac{x_1 + \cdots + x_n + mx_o}{n + m}\right) \le \frac{1}{n + m}\{f(x_1) + \cdots + f(x_n) + mf(x_o)\}.$$

If $x_o = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$, then x_o clearly lies in I, whence

$$f\left(\frac{x_1 + \cdots + x_n + \frac{m}{n}(x_1 + \cdots + x_n)}{m + n}\right) \le \frac{1}{m + n}[f(x_1 + \cdots + f(x_n)]$$

$$+ \frac{m}{m + n} f\left(\frac{x_1 + \cdots + x_n}{n}\right),$$

----

* The form of the principle that we use for the proof, sometimes referred to as proof by backward induction, proceeds by showing first that a sequence of propositions P_n is true for infinitely many indices n and then that the truth of P_n implies the truth of P_{n-1}. We observe that we have already established the truth of (5.3) for infinitely many values of n, namely, the powers of 2.

or

$$\left(1 - \frac{m}{n + m}\right) f\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq \frac{1}{n + m} [f(x_1) + \cdots + f(x_n)] ,$$

which reduces to (5.3).

Now (5.2) follows easily for any rational value of α in $0 \leq \alpha \leq 1$; indeed, if $\alpha = r/s$, where r and s are positive integers with $r \leq s$, then

$$\begin{aligned} f\left(\frac{r}{s} x_1 + (1 - \frac{r}{s})x_2\right) &= f\left(\frac{rx_1 + (s - r)x_2}{s}\right) \\ &= f\left(\frac{x_1 + \cdots + x_1 + x_2 + \cdots + x_2}{s}\right) \\ &\leq \frac{r}{s} f(x_1) + \frac{(s - r)}{s} f(x_2) = \alpha f(x_1) + (1 - \alpha)f(x_2). \end{aligned}$$

We remark that we have not yet used the continuity of $f(x)$. Now let α be an arbitrary real number with $0 \leq \alpha \leq 1$, and let x_1 and x_2 be any two points in I. Let us assume that there exists a value α_o with $0 < \alpha_o < 1$ and a pair of numbers x_1, x_2 in I such that (5.2) does not hold, i.e.,

$$(5.4) \qquad f(\alpha_o x_1 + (1 - \alpha_o)x_2) > \alpha_o f(x_1) + (1 - \alpha_o)f(x_2).$$

By continuity there must exist an interval J about the point $\underline{X}_o = \alpha_o x_1 + (1 - \alpha_o)x_2$ such that

$$(5.5) \qquad f(x) > \alpha_o f(x_1) + (1 - \alpha_o)f(x_2).$$

Let $d > 0$ be the difference between the two sides of (5.4), i.e.,

$$(5.6) \qquad f(\alpha_o x_1 + (1 - \alpha_o)x_2) - \alpha_o f(x_1) - (1 - \alpha_o)f(x_2) = d > 0.$$

We may find x_1' as close to x_1 as we please, x_2' as close to x_2 as we please, and rational α' as close to α_o as we please, so that

$$\underline{X}' = \alpha' x_1' + (1 - \alpha')x_2' = \alpha_o x_1 + (1 - \alpha_o)x_2 = \underline{X}_o ,$$

whence $f(\underline{X}') = f(\underline{X}_o)$. Now (5.2) with rational α' implies that

$$f(\underline{X}') \leq \alpha' f(x_1') + (1 - \alpha')f(x_2') \tag{5.7}$$

and (5.4) implies that

$$-f(\underline{X}_o) \leq -\alpha_o f(x_1) - (1 - \alpha_o)f(x_2),$$

whence

$$\begin{aligned} |f(\underline{X}') - f(\underline{X}_o')| &= |\alpha' f(x_1') + (1 - \alpha')f(x_2') - \alpha_o f(x_1) - (1 - \alpha_o)f(x_2)| \\ &\leq \alpha'|f(x_1') - f(x_1)| + (1 - \alpha')|f(x_2') - f(x_2)| + |\alpha' - \alpha_o|(|f(x_2)| + |f(x_1)|) \\ &\leq |f(x_1') - f(x_1)| + |f(x_2') - f(x_2)| + |\alpha' - \alpha_o|(|f(x_2)| + |f(x_1)|). \end{aligned}$$

Each of the three terms in the last line of this last inequality can be made less than d/6 by appropriate choices of x_1', x_2', and α'; this implies that (5.5) and (5.7) are in contradiction, so that (5.2) holds for arbitrary real α in $0 \leq \alpha \leq 1$.

Let us say briefly what we have shown. The definition (5.1) of convexity does not include the assumption of continuity; it says merely that the value of the function at the midpoint of any two points x_1 and x_2 cannot exceed the average of the values of the function at the points x_1 and x_2. The property (5.2), on the other hand, requires that the entire chord between $(x_1, f(x_1))$ and $(x_2, f(x_2))$ lie above the graph of the curve between x_1 and x_2 if α in (5.2) assumes all real values in $0 \leq \alpha \leq 1$. In this latter case, (5.1) is a simple consequence of (5.2); indeed, the continuity of f(x) is also an elementary consequence. However, without the continuity of f(x),

we can prove no more than (5.2) for rational α; (5.3) is a special case of this result.

Let us proceed directly to the principal inequality of this chapter.

THEOREM 5.2. (Jensen's Inequality) Let $f(x)$ be a continuous function defined in the interval I: $a \leq x \leq b$, and let $f(x)$ satisfy (5.1) in I. Then, for any $x_1, x_2, \cdots, x_n$ in I and for any positive numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that $\sum_{i=1}^{n} \alpha_i = 1$,

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i f(x_i), \tag{5.8}$$

with equality in (5.8) if $x_1 = x_2 = \cdots = x_n$, or if $f(x)$ reduces to a linear function.

Proof. Let us show that (5.8) reduces to an equality in the case that $f(x)$ is linear on I, or more precisely, in the case that $f(x)$ is linear between the smallest and largest values of the $x_1, \cdots, x_n$. Indeed, for $g(x) = Ax + B$,

$$g(\Sigma\alpha_i x_i) = A(\Sigma\alpha_i x_i) + B = A(\Sigma\alpha_i x_i) + B(\Sigma\alpha_i) = \Sigma\alpha_i(Ax_i + B) = \Sigma\alpha_i g(x_i), \tag{5.9}$$

which shows that equality prevails in (5.8) whenever $f(x)$ is the linear function $g(x)$.

To prove (5.8) in the case that $f(x)$ is not a linear function, we proceed by induction: Let us assume that (5.8) holds for $n = 1, 2, \cdots, k$, and let us consider $k + 1$ points $x_1, \cdots, x_k, x_{k+1}$ on I and $k + 1$ positive numbers $\alpha_1, \cdots, \alpha_k, \alpha_{k+1}$ such that $\alpha_1 + \cdots + \alpha_k + \alpha_{k+1} = 1$. We have

$$\sum_{i=1}^{k+1} \alpha_i \, f(x_i) = (1 - \alpha_{k+1}) \sum_{i=1}^{k} \frac{\alpha_i}{1 - \alpha_{k+1}} f(x_i) + \alpha_{k+1} f(x_{k+1}).$$

Since $\sum_{i=1}^{n} \alpha_i/(1 - \alpha_{k+1}) = 1$, it follows from the induction hypothesis that

$$\sum_{i=1}^{k} \frac{\alpha_i}{1 - \alpha_{k+1}} f(x_i) \geq f\left(\sum_{i=1}^{k} \frac{\alpha_i}{1 - \alpha_{k+1}} x_i\right), \tag{5.10}$$

whence, if we use (5.2),

$$\begin{aligned} \sum_{i=1}^{k+1} \alpha_i f(x_i) &\geq (1 - \alpha_{k+1}) f\left(\sum_{i=1}^{k} \frac{\alpha_i}{1 - \alpha_{k+1}} x_i\right) + \alpha_{k+1} f(x_{k+1}) \\ &\geq f\left(\sum_{i=1}^{k+1} \alpha_i x_i\right). \end{aligned} \tag{5.11}$$

Since (5.8) holds for $n = 1$ and $n = 2$, (5.8) now follows for all n. If all the x_i are equal, then equality clearly holds in (5.8). If, on the other hand, equality holds in (5.8), then, if we retrace our induction steps, we have equality $x_1 = \cdots = x_k$ up to (5.10). From (5.10) we then have

$$x_{k+1} = \sum_{i=1}^{k} \frac{\alpha_i}{1 - \alpha_{k+1}} x_i = x_k ,$$

whence it follows that $x_i = \cdots = x_k = x_{k+1}$.

5.3. Connections with Earlier Inequalities. The form of Jensen's inequality given by (5.8) allows us another unified approach to most of the inequalities of the preceding chapters. We have shown, to a great extent, that the inequality between the arithmetic and geometric means may be used

to derive many of the principal inequalities. As we mentioned in the Introduction, it is not wise to abandon the many approaches to a given mathematical problem or concept, for one then loses both intuition and insight, and, in the case of the elementary inequalities treated in this book, the interplay among the various inequalities.

In this section we shall show, mostly in the form of exercises, how widely Jensen's inequality may be used.

Example 6. The function $f(x) = |x|$ is convex on the interval $I: -\infty < x < \infty$, and its graph consists of two linear pieces, $y = x$ for $0 \leq x < \infty$ and $y = -x$ for $-\infty < x \leq 0$. Jensen's inequality (5.8), or in the simplest form (5.3), shows that

$$\left|\sum_{i=1}^{n} \alpha_i x_i\right| \leq \sum_{i=1}^{n} \alpha_i |x_i|,$$

or

$$\left|\sum_{i=1}^{n} x_i\right| \leq \sum_{i=1}^{n} |x_i| .$$

Equality holds only if all the x_i are the same, or, and this shows the need for the condition of linearity, if all the x_i lie on one or the other of the two linear pieces of the graph of $|x|$.

Example 7. For the function $f(x) = x^p$ for $0 \leq x < \infty$, $p > 0$, we must distinguish two cases, $p > 1$ where $f(x)$ is convex, and p in the range $0 < p < 1$ where $f(x)$ is concave (if $p = 1$, $f(x)$ is linear and we have nothing to prove). The simple form (5.3) of Jensen's inequality implies that

(5.12) $$\left(\frac{x_1 + \cdots + x_n}{n}\right)^p \le \frac{1}{n}\left(x_1^p + \cdots + x_n^p\right), \qquad p > 1,$$

and

(5.13) $$\left(\frac{x_1 + \cdots + x_n}{n}\right)^p \ge \frac{1}{n}\left(x_1^p + \cdots + x_n^p\right), \qquad 0 < p < 1,$$

with equality only if $x_1 = \cdots = x_n$; this is a stronger result than that of Exercise 1.25. For negative values of p with $x > 0$, the function $F(x) = x^p$ is convex, and (5.3) implies that

(5.14) $$\left(\frac{x_1 + \cdots + x_n}{n}\right)^p \le \frac{1}{n}\left(x_1^p + \cdots + x_n^p\right), \qquad p < 0,$$

with equality only if $x_1 = \cdots = x_n$. For $p = -1$, (5.14) gives us the inequality between the arithmetic and harmonic means discussed in Chapter 2, Section 3. If we apply the full force of Jensen's inequality (5.8), we have the following "weighted" forms of the last three inequalities for positive α_i with $\sum_{i=1}^{n} \alpha_i = 1$:

(5.15) $$\left(\sum_{i=1}^{n} \alpha_i x_i\right)^p \le \sum_{i=1}^{n} \alpha_i x_i^p , \qquad p > 0, \quad x_i \ge 0;$$

(5.16) $$\left(\sum_{i=1}^{n} \alpha_i x_i\right)^p \ge \sum_{i=1}^{n} \alpha_i x_i^p , \qquad 0 < p < 1, \quad x_i > 0;$$

(5.17) $$\left(\sum_{i=1}^{n} \alpha_i x_i\right)^p \le \sum_{i=1}^{n} \alpha_i x_i^p , \qquad p < 0, \quad x_i > 0,$$

with equality only if $x_1 = \cdots = x_n$.

Exercises

5.1. Show that the function $f(x) = \log x$ is concave on $0 < x < \infty$, and use (5.3), as modified for a concave function, to prove the inequality between the arithmetic and geometric means:

$$(x_1 x_2 \cdots x_n)^{1/n} \le \frac{x_1 + x_2 + \cdots + x_n}{n}. \tag{5.18}$$

5.2. If $\alpha_1, \cdots, \alpha_n$ are positive numbers such that $\sum_{i=1}^{n} \alpha_i = 1$, and if $x_1, \cdots, x_n$ are positive numbers, show that (5.8), applied to $\log x$, yields

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \le \alpha_1 x_1 + \alpha_2 x_2 + \cdots \alpha_n x_n. \tag{5.19}$$

(This inequality is frequently called the weighted form of the inequality between the arithmetic and geometric means.)

5.3. If x and y are positive, and if p and q are positive numbers such that $1/p + 1/q = 1$, show that (5.19) in Problem 5.2 implies (4.2) of Chapter 4, namely,

$$xy \le \frac{x^p}{p} + \frac{y^q}{q}, \tag{5.20}$$

which was the principal tool in the proof of Hölder's inequality.

5.4. Extend the result of Problem 5.3 to the case of n positive numbers $x_1, \cdots, x_n$ and n positive numbers $p_1, \cdots, p_n$ such that $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, so that

$$x_1 x_2 \cdots x_n \le \frac{x_1^{p_1}}{p_1} + \frac{x_2^{p_2}}{p_2} + \cdots + \frac{x_n^{p_n}}{p_n}. \tag{5.21}$$

5.5. Use the fact that x log x is convex for $x > 0$ to show that, for $\alpha > 0$, $\beta > 0$,

$$\left(\frac{\alpha + \beta}{2}\right)^{\alpha+\beta} \leq \alpha^{\alpha}\beta^{\beta}.$$

5.6. Extend the result of the preceding problem to the following: Let $x_1, \cdots, x_n$ be positive numbers, let $p_1, \cdots, p_n$ be positive numbers such that $\Sigma p_i = 1$. Show that

$$(p_1x_1 + \cdots + p_nx_n)^{p_1x_1+\cdots+p_nx_n} \leq x_1^{p_1x_1} \cdots x_n^{p_nx_n}. \tag{5.22}$$

5.7. If $\alpha > 1$ and $\beta > 1$, show that

$$\alpha^{\sqrt{\alpha}}\,\beta^{\sqrt{\beta}} \leq \left(\frac{\alpha + \beta}{2}\right)^{\sqrt{2\alpha+2\beta}}.$$

5.8. Let $x_1, x_2, \cdots, x_n$ be positive numbers less than π. Show that

$$\sin x_1 \sin x_2 \cdots \sin x_n \leq \sin^n\left(\frac{x_1 + \cdots + x_n}{n}\right).$$

5.9. Extend the inequality of Problem 5.8 to the following result. Let $f(x)$ be positive and concave on the interval I: $a \leq x \leq b$, and let $x_1, \cdots, x_n$ be n points on I. Show that

$$f(x_1)f(x_2) \cdots f(x_n) \leq \left[f\left(\frac{x_1 + \cdots + x_n}{n}\right)\right]^n,$$

or, what is the same thing, that the geometric mean of the values $f(x_1), \cdots, f(x_n)$ does not exceed the functional value of the arithmetic mean of $x_1, \cdots, x_n$. Can this inequality be further extended?

5.10. For $p > 1$ and $q > 1$ with $1/p + 1/q = 1$, show that $g(y) = y^{q-1}$ is the inverse function of $f(x) = x^{p-1}$ for $x \geq 0$, $y \geq 0$. If A_1 is the area between the graph of the curve and the x-axis between $0 \leq x \leq a$, and if A_2 is the area between the graph of the curve and the y-axis between $0 \leq y \leq b$, show that

$$ab \leq A_1 + A_2 = \frac{a^p}{p} + \frac{b^q}{q},$$

with equality if and only if $a^p = b^q$. Note that this gives another proof of (4.2) in Chapter 4.

5.11. Develop the idea of Problem 5.10 in the following way. Let $y = f(x)$, $f(0) = 0$, be a continuous increasing function of x for $x \geq 0$, that is, $f(x_1) < f(x_2)$ for $x_1 < x_2$, so that $f(x)$ has an inverse function $x = g(y)$, $g(0) = 0$, defined on some interval $0 < y < y_o$, where y_o may be $+\infty$. Define A_1 and A_2 as $A_1 = \int_o^a f(x)dx$, $A_2 = \int_o^a g(y)dy$, and prove that

$$ab \leq A_1 + A_2 = \int_o^a f(x)dx + \int_o^b g(y)dy,$$

with equality only if $b = f(a)$. The result is often called <u>Young's inequality</u>.

5.12. Show that, for $0 \leq x \leq 1$,

$$\frac{\pi}{2} x \leq x \sin^{-1} x + \sqrt{1 - x^2} .$$

5.13. Let $x = g(y)$ be the inverse function of $y = x^5 + x$ on $x \geq 0$. Evaluate the integral $\int_o^2 g(y)dy$.

5.14. Use the fact that $f(x) = \log(1 + x)$ is concave on $x \geq 0$ to show that, for positive numbers $a_1, a_2, \cdots, a_n$,

$$(1 + A)^n \geq (1 + a_1)(1 + a_2) \cdots (1 + a_n),$$

where A is the arithmetic mean of the numbers $a_1, a_2, \cdots, a_n$.

5.15. Use the fact that $f(x) = \log(1 + e^x)$ is convex on $-\infty < x < \infty$ to show that, for positive numbers $a_1, a_2, \cdots, a_n$,

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq (1 + g)^n,$$

where g is the geometric mean of the numbers $a_1, a_2, \cdots, a_n$.

5.16. Let $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ be two sets of positive numbers. Show that

$$\prod_{k=1}^{n} a_k^{1/n} + \prod_{k=1}^{n} b_k^{1/n} \leq \prod_{k=1}^{n} (a_k + b_k)^{1/n}.$$

5.17. Let $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ be two sets of positive numbers, and let g_a be the geometric mean of $a_1, \cdots, a_n$ and g_b the geometric mean of $b_1, \cdots, b_n$. Show that

$$(g_a + g_b)^n \leq (a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n).$$

5.18. Let $y = f(x)$, $f(0) = 0$, be a continuous increasing function of x for $x \geq 0$, and let $x = g(y)$ be the uniquely defined inverse function of $f(x)$, $0 \leq y < y_o \leq +\infty$. Let $F(x,y)$, $p(x,y)$, and $q(x,y)$ be non-negative functions defined for $x \geq 0$, $y \geq 0$, such that $dF = pdx + qdy$. Show that, if $F(0,0) = 0$ and $F_{xy} \geq 0$, then

$$F(a,b) \leq \int_0^a p(x,f(x))dx + \int_0^b q(g(y),y)dy.$$

5.19. Let $y = f(x)$, $f(0) = 0$, be a continuous increasing function of x for $x \geq 0$, and let $x = g(y)$ be the inverse function of $f(x)$, as described in Problem 5.11. Rotate the curve $y = f(x)$ between $x = 0$ and $x = a$ to obtain a solid of rotation of volume V_x and let $\bar{x}$ be the x-coördinate of its centroid; and rotate $x = g(y)$ about the y-axis between $y = 0$ and $y = b$ to obtain a solid of rotation of volume V_y whose centroid is $\bar{y}$. Show that

$$\frac{\pi}{2} a^2 b^2 \leq \bar{x}\, V_x + \bar{y}\, V_y.$$

5.4. Inequalities for a triangle. One of the most interesting elementary applications of Jensen's theorem is the derivation of certain inequalities in the case that x_1, x_2, x_3 are the measures of the angles of a triangle. We avoid the use of the words "triangle inequalities", which are used in another context in mathematics.

Example 8. If x_1, x_2, x_3 are the angles of a triangle, examine the expression $\sin x_1 + \sin x_2 + \sin x_3$ for possible maxima or minima. Now we know that $\sin x$ is concave for $0 < x < \pi$, so that

$$\sin\left(\frac{x_1 + x_2 + x_3}{3}\right) \geq \frac{1}{3}(\sin x_1 + \sin x_2 + \sin x_3).$$

Hence, $\sin x_1 + \sin x_2 + \sin x_3 \leq 3 \sin\left(\frac{x_1 + x_2 + x_3}{3}\right)$, with equality only if $x_1 = x_2 = x_3$, i.e., only in the case of an equilateral triangle. But $x_1 + x_2 + x_3 = \pi$, so that equality--the case of a maximum--gives the value $3 \sin \pi/3 = 3\sqrt{3}/2$.

Example 9. If x_1, x_2, x_3 are the angles of a triangle, examine the expression $\sin x_1 \sin x_2 \sin x_3$ for possible maxima or minima. Since $\sin x$ is concave, we may apply either Problem 5.8 or its extension, Problem 5.9, to obtain

$$\sin x_1 \sin x_2 \sin x_3 \leq \sin^3 \frac{x_1 + x_2 + x_3}{3} = \sin^3 \frac{\pi}{3} = 3\sqrt{3}/8 ,$$

with equality, and therefore a maximum, whenever $x_1 = x_2 = x_3$, which is again the case of an equilateral triangle.

In problems of this kind, it frequently happens that one must restrict the range of the angles involved; we see this occurrence in the next example.

Example 10. If x_1, x_2, x_3 are the angles of a triangle, examine the expression $\tan x_1 + \tan x_2 + \tan x_3$ for possible maxima or minima. We point out that $\tan x$ is convex only on the interval $0 < x < \pi/2$, so that, in order to restrict x_1, x_2, x_3 to this interval, we assume at the outset that the triangle is acute. By (5.3) we have

$$\tan \frac{x_1 + x_2 + x_3}{3} \leq \frac{1}{3}(\tan x_1 + \tan x_2 + \tan x_3)$$

whence

$$3 \tan \frac{\pi}{3} \leq \tan x_1 + \tan x_2 + \tan x_3,$$

with equality only for $x_1 = x_2 = x_3$, i.e., only for the equilateral triangle, in which case the expression $\tan x_1 + \tan x_2 + \tan x_3$ has a minimum value $3\sqrt{3}$.

Exercises

5.20. Let x_1, x_2, x_3 denote the angles of a triangle. Find the maximum or minimum of the expression given, and indicate what further restrictions may be necessary.

(a) $\cos x_1 + \cos x_2 + \cos x_3$.

(b) $\cos x_1 \cos x_2 \cos x_3$.

(c) $\tan x_1 \tan x_2 \tan x_3$.

(d) $\sec x_1 + \sec x_2 + \sec x_3$.

(e) $\tan x_1/2 + \tan x_2/2 + \tan x_3/2$.

5.5. Further Developments. We wish to point out some geometrical properties of continuous convex functions which reinforce our intuition about such functions. The results we obtain, as well as the methods we use, are typical of the way in which real functions of one variable are treated.

We return to the geometry of Figure 5.1 and let x_1, x_2, x_3 be any three points of I such that $x_1 < x_2 < x_3$. If we use the identity

$$x_2 = \frac{x_3 - x_2}{x_3 - x_1} x_1 + \frac{x_2 - x_1}{x_3 - x_1} x_3 \equiv \alpha_1 x_1 + \alpha_2 x_3 ,$$

where $\alpha_1 + \alpha_2 = 1$, then Jensen's inequality (5.2) for a continuous convex function f(x) yields

$$f(x_2) \leq \frac{x_3 - x_2}{x_3 - x_1} f(x_1) + \frac{x_2 - x_1}{x_3 - x_1} f(x_3) ,$$

or, after multiplying by $x_3 - x_1$,

$$(x_3 - x_1)f(x_2) \leq (x_3 - x_2)f(x_1) + (x_2 - x_1)f(x_3). \tag{5.23}$$

If we subtract $(x_3 - x_1)f(x_1)$ from both sides of (5.23), we have

$$(x_3 - x_1)(f(x_2) - f(x_1)) \leq (x_2 - x_1)(f(x_3) - f(x_1)),$$

or

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1}.$$

If now we subtract $(x_3 - x_1)f(x_3)$ from both sides of (5.23), we obtain

$$(x_3 - x_1)(f(x_2) - f(x_3)) \leq (x_2 - x_3)(f(x_3) - f(x_1)),$$

or

$$\frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2},$$

and if we combine the two inequalities involving quotients, we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}. \tag{5.24}$$

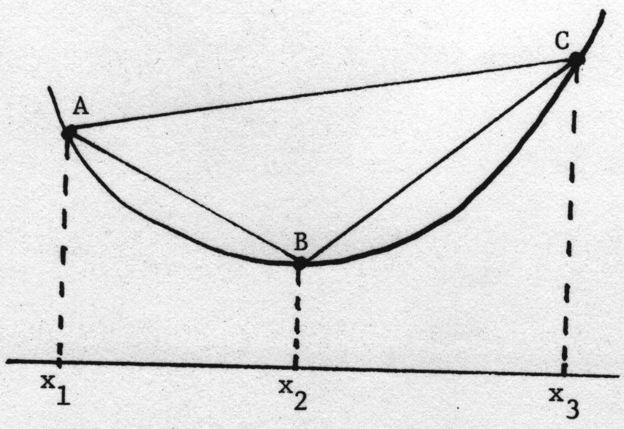


Fig. 5.5

If we interpret (5.24) geometrically, we see (Figure 5.5), that (5.24) is nothing more than the statement that

slope AB $\leq$ slope AC $\leq$ slope BC,

so that, as x_2 increases with x_1 held fixed, say, the slope of the chord AB is monotonically increasing. Now we know from elementary

analysis that a bounded, monotonically increasing function must tend to a limit. Let us write x for x_2 and $\Delta x = x_2 - x_1 > 0$; then (5.24) may be written as

$$\frac{f(x) - f(x - \Delta x)}{\Delta x} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1},$$

where the left-hand side is, as we have remarked, a monotonically increasing function of Δx which is bounded by the fixed quantity on the right-hand side. Hence

$$\lim_{\Delta x \to 0} \frac{f(x) - f(x - \Delta x)}{\Delta x} \qquad (\Delta x > 0)$$

exists at the point x; this is the so-called <u>left-hand</u> <u>derivative</u> $f'_-(x)$ of $f(x)$ at x. In a similar fashion we may show that the limit

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \qquad (\Delta x > 0)$$

exists at the point x; this is the so-called <u>right-hand</u> <u>derivative</u> $f'_+(x)$ of $f(x)$ at x. Thus we have proved the following result.

THEOREM 5.3. <u>A continuous convex function</u> $f(x)$ <u>in an interval</u> $I: a \leq x \leq b$ <u>has a left-hand derivative</u> $f'_-(x)$ <u>and a right-hand derivative</u> $f'_+(x)$ <u>at each point of</u> $a < x < b$. <u>Furthermore</u>, $f'_-(x) \leq f'_+(x)$.

We remark that if, at a point x in $a < x < b$, $f'_-(x) = f'_+(x)$, then $f(x)$ is differentiable at x, with $f'(x) = f'_-(x) = f'_+(x)$.

Our observation concerning the slopes of the chords in Figure 5.5 suggests that we may obtain even more information from (5.24). Let $x_1 < x_2$; then for sufficiently small $\Delta x > 0$ we have $x_1 + \Delta x < x_2 - \Delta x$, so that from (5.24) we have

$$\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \le \frac{f(x_2) - f(x_2 - \Delta x)}{\Delta x}.$$

In the limit as $\Delta x \to 0$ we then have

$$f'_+(x_1) \le f'_-(x_2),$$

and, since $f'_-(x_2) \le f'_+(x_2)$, we have $f'_+(x_1) \le f'_+(x_2)$ whenever $x_1 < x_2$, which shows that

> The right-hand derivative $f'_+(x)$ is a monotonically non-decreasing function of x in $a < x < b$; the same is true of the left-hand derivative $f'_-(x)$.

We summarize some of the facts obtained in the discussion above in the form of a theorem.

THEOREM 5.4. *The graph of a continuous function $f(x)$ satisfying (5.1) in I: $a \le x \le b$ possesses a line of support at each point x_o in I; that is, a line L: $y - f(x_o) = m(x - x_o)$ such that no point of the graph lies below L.*

The slope m of the line L of Theorem 5.4 may be taken, for the sake of definiteness, to be $\frac{1}{2}[f'_+(x_o) + f'_-(x_o)]$, so that m will coincide with the usual derivative $f'(x_o)$ whenever that derivative exists (see Problem 5.23).

Exercises

5.21. Show that a necessary and sufficient condition that a continuous function $f(x)$ be convex on I is that, for any three points x_1, x_2, x_3 in the interval I: $a \le x \le b$ such that $x_1 < x_2 < x_3$,

$$\begin{vmatrix} x_1 & f(x_1) & 1 \\ x_2 & f(x_2) & 1 \\ x_3 & f(x_3) & 1 \end{vmatrix} \geq 0.$$

5.22. Prove that the right-hand and left-hand derivatives $f'_+(x)$ and $f'_-(x)$ used in this section have the properties that $f'_+(x)$ is continuous from the right and that $f'_-(x)$ is continuous from the left.

5.23. Show that the set of points x on I where $f'_-(x) \neq f'_+(x)$ is denumerable.

We present next a classical inequality which has proved to be very useful in many contexts. We shall show that its interpretation in the context of the theory of convex--and concave--functions can lead to a powerful generalization of Jensen's theorem.

Let f(x) be continuous and non-negative for $a \leq x \leq b$. Then

$$(5.25) \qquad \frac{1}{b-a}\int_a^b \log f(x)dx \leq \log\left(\frac{1}{b-a}\int_a^b f(x)dx\right),$$

with equality only for f(x) constant. We reproduce the simple classical proof in order to analyze it from the point of view of the theory of convex functions. First, we set $m = \frac{1}{b-a}\int_a^b f(x)dx$, and then write $g(x) = f(x) - m$, so that $\int_a^b g(x) = 0$. Then

$$\frac{1}{b-a}\int_a^b \log f(x)dx = \frac{1}{b-a}\int_a^b \log(m + g(x))dx$$

$$= \frac{1}{b-a}\int_a^b [\log m + \log(1 + \frac{g(x)}{m})]dx.$$

If we apply to the last term the inequality

$$\log(1 + t) \leq t, \tag{5.26}$$

with equality only for $t = 0$, we have

$$\frac{1}{b - a}\int_a^b \log f(x)dx \leq \log m + \frac{1}{b - a}\int_a^b \frac{g(x)}{m}\,dx$$

$$= \log m = \log\left(\frac{1}{b - a}\int_a^b f(x)dx\right) ,$$

since $\int_a^b g(x)dx = 0$. Equality holds only if $g(x) = 0$, that is, only if $f(x)$ is the constant m.

The analysis of the proof turns on the simple inequality (5.26), and we observe that (5.26) is nothing more than the fact that the concave function $y = \log(1 + t)$ has a line of support at the point $t = 0$; since $\log(1 + t)$ has derivative 1 at $t = 0$, the line of support is simply the tangent line $y = t$, and the inequality $\log(1 + t) \leq t$ is simply the assertion that the curve $y = \log(1 + t)$ lies below the line of support $y = t$.

Suppose now that we have a continuous function $f(x)$ on $a \leq x \leq b$ and a continuous concave function $F(y)$ defined on the range of values assumed by $y = f(x)$ on $a \leq x \leq b$. Let $m = (b - a)^{-1} \int_a^b f(x)dx$ and let $g(x) = f(x) - m$, so that $\int_a^b g(x)dx = 0$. Since $F(y)$ is concave, it will have a line of support at $y = m$, that is, there will exist a number M such that, for all y described above,

$$F(y) \leq F(m) + M(y - m). \tag{5.27}$$

If we integrate (5.27) and divide by $b - a$, we have

$$\frac{1}{b-a}\int_a^b F(f(x))dx \le F(m) + \frac{M}{b-a}\int_a^b (f(x) - m)dx$$

$$= F(m) + \frac{M}{b-a}\int_a^b g(x)dx = F(m).$$

We state our result, which is an integral form of Jensen's theorem, in the form of a theorem.

THEOREM 5.5. *Let* $f(x)$ *be continuous on* $a \le x \le b$, *and let* F *be a continuous concave function, such that* $F(f(x))$ *is defined on* $a \le x \le b$. *Then*

$$\frac{1}{b-a}\int_a^b F(f(x))dx \le F\left(\frac{1}{b-a}\int_a^b f(x)dx\right). \tag{5.28}$$

If F *is convex, then*

$$\frac{1}{b-a}\int_a^b F(f(x))dx \ge F\left(\frac{1}{b-a}\int_a^b f(x)dx\right). \tag{5.29}$$

Equality holds only if $f(x)$ *is constant.*

Exercises

5.24. Let $f(x,y)$ be continuous and positive in the disc D_r: $(x - x_o)^2 + (y - y_o)^2 \le r^2$. Show that

$$\frac{1}{\pi r^2}\iint_{D_r} \log f(x,y)dxdy \le \log\left(\frac{1}{\pi r^2}\iint_{D_r} f(x,y)dxdy\right).$$

SOLUTIONS

Chapter I

1.1. If we divide both sides of (1.2) by 2 and add $(a^2 + b^2)/4$ to both sides, we obtain

$$\frac{a^2 + b^2}{2} \ge \frac{a^2 + b^2}{4} + \frac{ab}{2} = \left(\frac{a + b}{2}\right)^2,$$

which, upon taking the square root, yields (1.3). Since equality holds in (1.2) if and only if $a = b$, the same is true in (1.3).

1.2. (a) We have

$$\begin{cases} (2a - b - c)^2 \ge 0 \\ (2b - c - a)^2 \ge 0 \\ (2c - a - b)^2 \ge 0 \,, \end{cases}$$

with equality holding everywhere if and only if $2a = b + c$, $2b = c + a$, $2c = a + b$, or $a = b = c$. If we expand these inequalities, we obtain three inequalities of the form

$$4a^2 + b^2 + c^2 \ge 4ab + 4ca - 2bc.$$

Then adding the three inequalities thus obtained and dividing by 6 gives (1.5).

(b) We have

$$\begin{aligned} a^2 - 2ac + c^2 = (a - c)^2 &\ge \left(\frac{a - c}{2}\right)^2 \\ &= \left[\frac{(a - b) + (b - c)}{2}\right]^2 \ge (a - b)(b - c) \\ &= ab + bc - ac - b^2, \end{aligned}$$

or

$$a^2 + b^2 + c^2 \ge ab + bc + ca,$$

with equality if and only if $a - c = 0$ and $a - b = b - c$, or $a = b = c$.

1.3. By (1.6) and (1.1), we have

$$\begin{aligned} a^3 + b^3 + c^3 &\geq ab\left(\frac{a+b}{2}\right) + bc\left(\frac{b+c}{2}\right) + ca\left(\frac{c+a}{2}\right) \\ &\geq ab\sqrt{ab} + bc\sqrt{bc} + ca\sqrt{ca} \\ &= (bc)^{3/2} + (ca)^{3/2} + (ab)^{3/2}, \end{aligned}$$

with equality if and only if $a = b = c$.

Alternatively, simply substituting $a^{3/2}$, $b^{3/2}$, $c^{3/2}$ for a,b,c in (1.5) gives the desired inequality, which reduces to equality if and only if $a^{3/2} = b^{3/2} = c^{3/2}$, or $a = b = c$.

1.4. The proof is straightforward from the hint. Equality holds if and only if $a = b = c$.

1.5. If we multiply the three inequalities

$$a + b \geq 2\sqrt{ab}, \quad b + c \geq 2\sqrt{bc}, \quad c + a \geq 2\sqrt{ca},$$

which come from (1.1), we obtain the desired result. Equality holds if and only if $a = b = c$.

1.6. First solution. From Exercise 1.4, we have

$$ab\left(\frac{a+b}{2}\right) + bc\left(\frac{b+c}{2}\right) + ca\left(\frac{c+a}{2}\right) \geq 3abc.$$

Multiplying this by $2/abc$ gives the desired result, with equality if and only if $a = b = c$.

Second solution. Notice that by (1.1), for all $x > 0$,

$$x + \frac{1}{x} \geq 2,$$

with equality if and only if x = 1. Therefore

$$\frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} = \left(\frac{a}{c} + \frac{c}{a}\right) + \left(\frac{b}{c} + \frac{c}{b}\right) + \left(\frac{b}{a} + \frac{a}{b}\right) \geq 6,$$

with equality if and only if $\frac{a}{c} = \frac{b}{c} = \frac{b}{a} = 1$, or $a = b = c$.

<u>Third Solution</u>. We shall use (1.1) and then the inequality deduced at the end of Problem 1.4. We have

$$\frac{a+b}{c} \quad \frac{b+c}{a} \quad \frac{c+a}{b} \geq 2\,\frac{\sqrt{ab}}{c} + 2\,\frac{\sqrt{bc}}{a} + 2\,\frac{\sqrt{ca}}{b}$$

$$\geq 2\cdot 3\left(\frac{\sqrt{ab}\ \sqrt{bc}\ \sqrt{ca}}{c\cdot a\cdot b}\right)^{1/3} = 6.$$

1.7. Multiply the first inequality in (1.4) by c^2, the second by a^2, and the third by b^2. Upon addition of these new inequalities and division by 2, the desired result is obtained. Equality holds if and only if $a = b = c$.

1.8. <u>First solution</u>. Division by abc of the inequality in Problem 1.7 gives this inequality.

<u>Second solution</u>. (see the second solution of Problem 1.6). We have

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} = \frac{1}{2}\left[\left(\frac{a}{c} + \frac{c}{a}\right)b + \left(\frac{b}{c} + \frac{c}{b}\right)a + \left(\frac{b}{a} + \frac{a}{b}\right)c\right] \geq a + b + c,$$

with equality if and only if $a = b = c$.

1.9. As we saw in Example 2, $a^3 + b^3 \geq ab(a + b)$, with equality if and only if $a = b$. If we multiply this by 3 and add $a^3 + b^3$, we obtain $4(a^3 + b^3) \geq (a + b)^3$, as desired.

<u>Remark</u>. We have shown that

$$a^3 + b^3 \geq \frac{(a+b)^3}{4} = \left(\frac{a+b}{2}\right)^2 (a + b) \geq ab(a + b),$$

using (1.1). Thus, we have ended up with an inequality stronger than the one we began with.

1.10. (a) By (1.1), we have

$$(1)\qquad \left[\frac{(a + c) + (b + d)}{2}\right]^2 \geq (a + c)(b + d) ,$$

with equality holding if and only if $a + c = b + d$. If we take the square root and divide by 2, we arrive at the desired inequality.

(b) In (1) above, we grouped $a + b + c + d$ as $(a + c) + (b + d)$. If we instead group it as $(a + b) + (c + d)$ or $(a + d) + (b + c)$ and use similar inequalities, then addition of these three inequalities gives

$$\frac{3}{4}(a + b + c + d)^2 \geq 2(ab + bc + cd + da + ac + bd),$$

with equality if and only if $a + c = b + d$, $a + b = c + d$, and $a + d = b + c$, or $a = b = c = d$. Now dividing by 12 and taking square roots gives the desired inequality.

1.11. Suppose that the sequences are increasing. Then $\{b_1c_1, b_2c_2\}$ is also increasing; note that positivity is necessary for this to hold generally. Thus (1.7) gives

$$\frac{a_1b_1c_1 + a_2b_2c_2}{2} \geq \frac{a_1 + a_2}{2} \cdot \frac{b_1c_1 + b_2c_2}{2} .$$

Applying (1.7) again and noting that $\frac{a_1 + a_2}{2}$ is positive, we get the desired inequality. Equality holds if and only if $a_1 = a_2$ or $b_1c_1 = b_2c_2$, and $b_1 = b_2$ or $c_1 = c_2$. It is easily seen that this is true if and only if <u>all</u> <u>but</u> <u>one</u> of the sequences $\{a_1, a_2\}$, $\{b_1, b_2\}$, $\{c_1, c_2\}$ is constant.

1.12. The generalization of Problem 1.11 is that for n increasing (or decreasing) positive sequences $\{a_{k1}, a_{k2}\}$, $k = 1, \cdots, n$, we have

$$\frac{a_{11}a_{21}\cdots a_{n1} + a_{12}a_{22}\cdots a_{n2}}{2} \geq \left(\frac{a_{11} + a_{12}}{2}\right)\left(\frac{a_{21} + a_{22}}{2}\right)\cdots\left(\frac{a_{n1} + a_{n2}}{2}\right).$$

with equality if and only if all but one of the sequences is constant (see Exercise 1.24 for a full proof, in fact, of a more general inequality). Therefore, if $a, b, \mu_1, \cdots, \mu_n$ are positive numbers, we have

$$\frac{a^{\mu_1 + \cdots + \mu_n} + b^{\mu_1 + \cdots + \mu_n}}{2} \geq \left(\frac{a^{\mu_1} + a^{\mu_2}}{2}\right) \cdots \left(\frac{a^{\mu_n} + b^{\mu_n}}{2}\right)$$

with equality if and only if $a = b$. Letting $\mu_1 = \cdots = \mu_n = 1$ and taking n^{th} roots gives

$$\left(\frac{a^n + b^n}{2}\right)^{1/n} \geq \frac{a + b}{2} .$$

1.13. Let us denote $\left(\frac{1 - t^2}{1 + t^2}\right)^2$ by a and $\left(\frac{2t}{1 + t^2}\right)^2$ by b. Then by the previous problem,

$$a^6 + b^6 \geq \frac{1}{2^5}(a + b)^6 = \frac{1}{32} ,$$

with equality if and only if $a = b$. In $0 \leq t \leq 1$, this is equivalent to $1 - t^2 = 2t$, or $t = \sqrt{2} - 1$.

1.14. It follows from Problem 1.12 that

$$\frac{a^{2n} + b^{2n}}{2} \geq \left(\frac{a^2 + b^2}{2}\right)^n = \frac{1}{2} .$$

with equality if and only if $a = b$.

1.15. Using Problem 1.12, we have

$$1 + \tan^8\theta = 1^4 + (\tan^2\theta)^4 > \frac{1}{2^3}(1 + \tan^2\theta)^4 = \frac{1}{8}\sec^8\theta.$$

except where $1 = \tan^2\theta$, or $\theta = \pi/4$.

1.16. In the second inequality of the solution to Exercise 1.12. let $n = 3$ and $\mu_1 = 1$, $\mu_2 = 2$, $\mu_3 = 3$. The desired result follows immediately. We have equality if and only if $a = b$.

1.17. Let $d = \sqrt{x^2 + y^2}$ be the distance from the origin to a point on the line. Then by (1.3),

$$d \geq \frac{x + y}{\sqrt{2}} = \frac{3}{\sqrt{2}},$$

with equality at $x = y = 3/2$. That is, the minimum distance is $3/\sqrt{2}$.

1.18. Let $d = \sqrt{x^2 + y^2}$ be the distance from the origin to a point on the arc. Then by using (1.3) twice,

$$d \geq \frac{x + y}{\sqrt{2}} \geq \frac{(\sqrt{x} + \sqrt{y})^2}{2\sqrt{2}} = \frac{a}{2\sqrt{2}}$$

with equality at $x = y = a/4$. The minimum distance is $a/2\sqrt{2}$.

1.19. Under the hypotheses, we have $(a_1 - a_2)(b_1 - b_2) \leq 0$, with equality if and only if $a_1 = a_2$ or $b_1 = b_2$. Proceeding with the argument in the text shows that we get (1.7) with the opposite inequality sign.

1.20. (a) If $d = \sqrt{x^2 + y^2 + z^2}$ is the distance from the origin to a point on the plane, then by (1.12) we have

$$d^2 \geq \frac{(x + y + z)^2}{3} = \frac{25}{3},$$

with equality at $x = y = z = 5/3$. The minimum distance is therefore $5/\sqrt{3}$.

(b) We now apply (1.12) twice:

$$d^2 \geq \frac{(x+y+z)^2}{3} \geq \frac{(\sqrt{x}+\sqrt{y}+\sqrt{z})^4}{27} = \frac{a^2}{27}.$$

The minimum distance is thus $a/3\sqrt{3}$, achieved at $x = y = z = a/9$.

1.21. The inequality (1.12) yields

$$f(x,y) \geq \frac{1}{3}\left[\sin^2 x + \left(\frac{1-y^2}{1+y^2}\right)^2 \cos^2 x + \left(\frac{2y}{1+y^2}\right)^2 \cos^2 x\right]^2$$
$$= \frac{1}{3}(\sin^2 x + \cos^2 x)^2 = \frac{1}{3},$$

with equality if and only if $\sin^2 x = \left(\frac{1-y^2}{1+y^2}\right)^2 \cos^2 x = \left(\frac{2y}{1+y^2}\right)^2 \cos^2 x$ or $\sin x = \frac{1}{\sqrt{3}}$, $\cos x = \sqrt{\frac{2}{3}}$, $y = \sqrt{2} - 1$.

1.22. If we apply (1.12), we obtain

$$f(x) \geq \frac{1}{3}(1^2 + \cos^2 x + \sin^2 x)^2 = 4/3.$$

However, 4/3 is not the minimum value of $f(x)$ since equality would require $1 = \cos x = \sin x$, which is impossible.

This example should serve to forewarn the reader that in looking for extrema by using inequalities, one must be sure that equality is actually attained.

1.23. Let $\{a_1, \cdots, a_n\}$ and $\{b_1, \cdots, b_n\}$ be two increasing sequences of real numbers. Then, if $i < j$,

(1) $$(a_i - a_j)(b_i - b_j) \geq 0,$$

or

$$a_i b_i + a_j b_j \geq a_i b_j + a_j b_i .$$

If we sum all such inequalities and then add $\sum_{i=1}^{n} a_i b_i$ to both sides of the resulting inequality, we obtain

$$n\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j = \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) .$$

Division by n^2 produces (1.11). Now if one of the sequences is constant, then equality clearly holds. Conversely, if equality holds in (1.11), then equality holds in (1) for all $i < j$. In particular, equality holds for $i = 1$ and $j = n$. Therefore $a_1 = a_n$ or $b_j = b_n$, but since each sequence is monotonic, one of the sequences must be constant.

If the two sequences were both decreasing, then (1) would still hold, so that (1.11) is valid. If one sequence is increasing while the other is decreasing, then the inequality in (1) is reversed, so that (1.11) is valid with the sense of inequality reversed.

1.24. We shall show that if $\{a_{i1}, \cdots, a_{in}\}$ are increasing (or else decreasing) sequences of positive numbers for $i = 1, \cdots, m$, then

(1) $$\frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{m} a_{ij} \geq \prod_{i=1}^{m}\frac{1}{n}\sum_{j=1}^{n} a_{ij}$$

with equality if and only if all but one of the sequences is constant.

We shall proceed by induction. We have seen (1) for $m = 2$ in (1.11). Thus, assume (1) for $m = p - 1$, $p \geq 3$, and let $\{a_{ij}\}_{j=1}^{n}$, $i = 1, \cdots, p$ be p positive increasing sequences. Since $\left\{\prod_{i=1}^{p-1} a_{ij}\right\}_{j=1}^{n}$ is an increasing sequence, (1.11) gives us

(2) $$\frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{p} a_{ij} \geq \left(\frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{p-1} a_{ij}\right)\left(\frac{1}{n}\sum_{j=1}^{n} a_{pj}\right),$$

with equality if and only if $\left\{\prod_{i=1}^{p-1} a_{ij}\right\}_{j=1}^{n}$ or $\{a_{pj}\}_{j=1}^{n}$ is constant. Applying the induction hypothesis and observing that $\sum_{j=1}^{n} a_{pj}$ is positive, we have

$$\left(\frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{p-1} a_{ij}\right)\left(\frac{1}{n}\sum_{j=1}^{n} a_{pj}\right) \geq \left(\prod_{i=1}^{p-1}\frac{1}{n}\sum_{j=1}^{n} a_{ij}\right)\left(\frac{1}{n}\sum_{j=1}^{n} a_{pj}\right)$$

$$= \prod_{i=1}^{p}\frac{1}{n}\sum_{j=1}^{n} a_{ij} ,$$

with equality if and only if all but one of the sequences $\{a_{ij}\}_{j=1}^{n}$, $i = 1, \cdots, p-1$, are constant. Combining this with (2) yields (1), together with the proper equality condition.

1.25. Without loss of generality, we may assume that $a_1 \leq a_2 \leq \cdots \leq a_n$ (otherwise, we simply relabel them and note that the result is symmetric). Then, by the generalization of Chebyshev's inequality given in Exercise 1.24,

$$\frac{\sum_{k=1}^{n} a_k^m}{n} \geq \left(\frac{\sum_{k=1}^{n} a_k}{n}\right)^m$$

where we have let all m sequences be the same. Thus the result follows.

<u>Remark</u>. This result will be generalized in Chapter 5.

1.26. <u>First solution</u>. Let $x \geq y$ and $z = y/x$. Then we wish to prove that $(1 + z^m)^n < (1 + z^n)^m$, or

$$\sum_{k=1}^{n}\binom{n}{k} z^{mk} < \sum_{k=1}^{m}\binom{m}{k} z^{nk} .$$

Not only does the right-hand side contain more terms than the left, but each term is also greater than the corresponding term on the left. Hence the inequality is true.

<u>Second solution</u>. Let t be such that $(\frac{x}{t})^n + (\frac{y}{t})^n = 1$. Then since $x,y > 0$, we have $(\frac{x}{t})^m < (\frac{x}{t})^n < 1$, $(\frac{y}{t})^m < (\frac{y}{t})^n < 1$, and

$$[(\frac{x}{t})^m + (\frac{y}{t})^m]^n < [(\frac{x}{t})^n + (\frac{y}{t})^n]^n = 1 = [(\frac{x}{t})^n + (\frac{y}{t})^n]^m.$$

Multiplying by t^{mn} gives the desired result.

1.27. Since $a + b \geq 2\sqrt{ab}$, with equality for $a = b$, we have $(1 + x_1)\cdots(1 + x_n) \geq (2\sqrt{x_1})\cdots(2\sqrt{x_n}) = 2^n$, with equality for $x_1 = \cdots = x_n = 1$.

1.28. By the same method as in the preceding problem we have

$$(x_1 + x_{p1})\cdots(x_n + x_{pn}) \geq (2\sqrt{x_1x_{p1}})\cdots(2\sqrt{x_nx_{pn}}) = 2^n.$$

Equality holds for $x_1 = \cdots = x_n = 1$.

1.29. (a) Let $x + y = c$, a constant. Then by (1.1), $xy \leq c^2/4$, with equality at $x = y$. In other words, xy is a maximum when $x = y$ $(= c/2)$.

(b) Let $xy = c$. Then by (1.1), $x + y \geq 2\sqrt{c}$, with equality for $x = y$. Thus the minimum value of $x + y$ is achieved when $x = y$ $(= \sqrt{c})$.

(c) The proof is the same as in part (a), except that we use the inequality $\frac{a + b + c}{3} \geq \sqrt[3]{abc}$ of Exercise 1.4 in place of (1.1).

Remark. Parts (a) and (b) are referred to as dual properties. There is an obvious dual to part (c).

1.30. To maximize $x\sqrt{16 - x^2}$ is to maximize its square, $x^2(16 - x^2)$. Regarding $x^2(16 - x^2)$ as a product of two factors whose sum is constant, 16, we see that the maximum occurs, by Exercise 1.29(a), when $x^2 = 16 - x^2$ or $x = 2\sqrt{2}$. The maximum value of $x\sqrt{16 - x^2}$ is then 8.

Alternatively, we could have used (1.1) directly:

$$x\sqrt{16 - x^2} = \sqrt{x^2(16 - x^2)} \le \frac{x^2 + (16 - x^2)}{2} = 8.$$

1.31. There are two ways of expressing the same solution, as in the previous exercise. Here, however, we present only one. By Problem 1.4, we have

$$xyz = \frac{1}{6}[x(2y)(3z)] \le \frac{1}{6}\left(\frac{x + 2y + 3z}{3}\right)^3 = \frac{4}{3},$$

with equality if and only if $x = 2y = 3z (= 2)$, or $x = 2$, $y = 1$, $z = 2/3$.

1.32. Let x,y,z be the lengths of the edges emanating from a fixed vertex. The volume is then xyz with $4(x + y + z)$ constant. By Problem 1.29(c), the volume is a maximum when $x = y = z$, i.e., when the figure is a cube.

1.33. If x,y,z are as in the previous problem, then the volume $V = xyz$ and the surface area is $2(xy + yz + zx)$. Those values of x,y,z which maximize V also maximize $V^2 = (xy)(yz)(zx)$. The sum of the three factors in V^2 being a constant, V^2 will be maximized whenever $xy = yz = zx$, which is equivalent to $x = y = z$. Hence the figure of maximum volume is a cube.

1.34. By Problem 1.25 with k = 4, we have

$$\left(\frac{x^2 + y^2 + z^2}{3}\right)^{1/4} \ge \frac{\sqrt{x} + \sqrt{y} + \sqrt{z}}{3},$$

or $\sqrt{x} + \sqrt{y} + \sqrt{z} \le 3^{3/4}$, with equality if and only if $x = y = z = 1/\sqrt{3}$.

1.35. Without loss of generality, we may assume $a_1 \ge a_2 \ge \cdots \ge a_n$. Hence $\frac{1}{a_1} \le \frac{1}{a_2} \le \cdots \le \frac{1}{a_n}$, and by Chebyshev's inequality, Problem 1.23, we have

$$\frac{a_1 + \cdots + a_n}{n} \cdot \frac{\frac{1}{a_1} + \cdots + \frac{1}{a_n}}{n} \ge \frac{a_1 \cdot \frac{1}{a_1} + \cdots + a_n \cdot \frac{1}{a_n}}{n} = 1,$$

with equality if and only if $a_1 = \cdots = a_n$.

1.36. First by (1.3),

$$(a + \tfrac{1}{a})^2 + (b + \tfrac{1}{b})^2 \ge \tfrac{1}{2}(a + \tfrac{1}{a} + b + \tfrac{1}{b})^2 = \tfrac{1}{2}(1 + \tfrac{1}{a} + \tfrac{1}{b})^2,$$

with equality if and only if $a + \frac{1}{a} = b + \frac{1}{b}$, or $(a - b)(1 - \frac{1}{ab}) = 0$, or $a = b$. By the previous exercise,

$$\frac{1}{a} + \frac{1}{b} \ge \frac{4}{a + b} = 4,$$

with equality, again, if and only if $a = b = \frac{1}{2}$. Therefore

$$(a + \tfrac{1}{a})^2 + (b + \tfrac{1}{b})^2 \ge \tfrac{1}{2}(1 + 4)^2 = \tfrac{25}{2}.$$

Remark. Though by (1.1) we have $a + \frac{1}{a} \ge 2$ and $b + \frac{1}{b} \ge 2$, or $(a + \frac{1}{a})^2 + (b + \frac{1}{b})^2 \ge 8$, the value 8 cannot be achieved since this would require $a = b = 1$. Recall Exercise 1.22.

1.37. Using the inequality of Problem 1.25 with k = 2, we have

$$\sum_{j=1}^{n}\left(a_j + \frac{1}{a_j}\right)^2 \geq \frac{1}{n}\left[\sum_{j=1}^{n}\left(a_j + \frac{1}{a_j}\right)\right]^2 = \frac{1}{n}\left[1 + \sum_{j=1}^{n}\frac{1}{a_j}\right]^2 ,$$

with equality if and only if $a_1 + \frac{1}{a_1} = \cdots = a_n + \frac{1}{a_n}$, or $a_1 = \cdots = a_n = \frac{1}{n}$. Now by Problem 1.35,

$$\sum_{j=1}^{n}\frac{1}{a_j} \geq \frac{n^2}{\sum_{j=1}^{n} a_j} = n^2,$$

with equality again at $a_1 = \cdots = a_n = 1/n$. Hence the result follows.

1.38. From Exercise 1.12,

$$(a + \tfrac{1}{a})^n + (b + \tfrac{1}{b})^n \geq \frac{1}{2^{n-1}}(a + b + \tfrac{1}{a} + \tfrac{1}{b})^n \geq \frac{5^n}{2^{n-1}} ,$$

as in Exercise 1.36, with equality if and only if $a = b = 1/2$. Therefore

$$\sum_{n=1}^{\infty}\frac{1}{(a + \frac{1}{a})^n + (b + \frac{1}{b})^n} \leq \sum_{n=1}^{\infty}\frac{2^{n-1}}{5^n} = \frac{1}{3} .$$

1.39. Since $a,b > 1$, both $\log_a b$ and $\log_b a$ are positive. Using the fact that $\log_b a = 1/\log_a b$, we have

$$\log_a b + \log_b a \geq 2,$$

with equality if and only if $\log_a b = 1$, or $a = b$. Therefore

$$\sum_{n=1}^{\infty}\frac{1}{(\log_a b + \log_b a)^n} \leq \sum_{n=1}^{\infty}\frac{1}{2^n} = 1.$$

1.40. We may assume without loss of generality that $a_1 \geq a_2 \geq \cdots \geq a_n$. Then $\{a_i^p\}_{i=1}^n$ is a decreasing sequence while $\{a_i^q\}_{i=1}^n$ is an increasing one. Therefore the result follows from Exercise 1.23, and equality holds if and only if $a_1 = \cdots = a_n$.

1.41. As in the last exercise, let $a_1 \geq a_2 \geq \cdots \geq a_n$. Then $\{a_i^p\}_{i=1}^n$ and $\{a_i^q\}_{i=1}^n$ are both increasing sequences, so that by (1.11), the result follows.

Remark. The same inequality holds for $p,q > 0$.

1.42. From (1.1) we have

$$\frac{bc}{b+c} \leq \frac{b+c}{4}, \quad \frac{ca}{c+a} \leq \frac{c+a}{4}, \quad \frac{ab}{a+b} \leq \frac{a+b}{4} .$$

Adding these three inequalities gives the desired inequality. Equality holds if and only if $a = b = c$.

1.43. Without loss of generality, we may assume that $a \leq b \leq c$. Then $\log a \leq \log b \leq \log c$, so that by (1.10)

$$a \log a + b \log b + c \log c \geq \frac{a+b+c}{3} (\log a + \log b + \log c),$$

or

$$\log(a^a b^b c^c) \geq \log(abc)^{\frac{a+b+c}{3}} ,$$

which gives the desired result. Equality holds if and only if $a = b = c$.

Remark. The inequality clearly generalizes to n numbers.

1.44. If we add $2(ab + bc + ca)$ to inequality (1.5), we obtain

$$(a + b + c)^2 \geq 3(ab + bc + ca),$$

or

$$\frac{a + b + c}{3} \geq \sqrt{\frac{ab + bc + ca}{3}}\ ,$$

with equality if and only if $a = b = c$. Now applying Exercise 1.4, we obtain

$$\sqrt{\frac{ab + bc + ca}{3}} \geq \sqrt{(ab \cdot bc \cdot ca)^{1/3}} = \sqrt[3]{abc}\ ,$$

with equality if and only if $ab = bc = ca$, or $a = b = c$.

CHAPTER II: SOLUTIONS

2.1. First solution. From (2.2), we have

$$\frac{1}{n}\left(\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_n}{a_1}\right) \geq \left(\frac{a_1}{a_2} \cdot \frac{a_2}{a_3} \cdot \cdots \cdot \frac{a_n}{a_1}\right)^{1/n} = 1,$$

which gives us the desired inequality. Equality holds if and only if $\frac{a_1}{a_2} = \frac{a_2}{a_3} = \cdots = \frac{a_n}{a_1} = t$, say. Then $t^n = \frac{a_1}{a_2} \cdot \frac{a_2}{a_3} \cdot \cdots \cdot \frac{a_n}{a_1} = 1$. Hence, $a_1 = a_2 = \cdot \cdots = a_n$.

Second solution. The inequality is clearly true for $n = 1$ and $n = 2$. Assume that it is true for $n = k$. We can assume, without loss of generality, that a_{k+1} is the smallest of the $a_1, a_2, \cdots, a_{k+1}$. By the inductive assumption,

(1) $$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_k}{a_1} \geq k.$$

We also have that

$$(a_1 - a_{k+1})(a_k - a_{k+1}) \geq 0,$$

or

(2) $$\frac{a_k}{a_{k+1}} + \frac{a_{k+1}}{a_1} - \frac{a_k}{a_1} \geq 1 .$$

Adding (1) and (2), we get the desired inequality.

Remark. The following theorem is clearly also true: Let $\{b_1, \cdots, b_n\}$ be a permutation of $\{a_1, \cdots, a_n\}$. Then $\sum_{i=1}^{n} \frac{a_i}{b_i} \geq n$, with equality if and only if $a_i = b_i$ for all i.

2.2. First solution. The simplest method of proof is to apply Chebyshev's inequality in the form (1.11) of Chapter I with $a_i = b_i$, $i = 1,2,\cdots,n$. Thus

$$\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n} \geq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^2,$$

which gives us (2.5), with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Second solution. Let us expand $(\sum_{i=1}^{n} a_i)^2$ and use the simple result that $2ab \leq a^2 + b^2$ (cf. (1.2), Chapter I):

$$\begin{aligned} \left(\sum_{i=1}^{n} a_i\right)^2 &= \sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j \\ &\leq \sum_{i=1}^{n} a_i^2 + \sum_{i<j} (a_i^2 + a_j^2) \\ &= \sum_{i=1}^{n} a_i^2 + (n-1)\sum_{i=1}^{n} a_i^2 = n\sum_{i=1}^{n} a_i^2 . \end{aligned}$$

Hence (2.5) follows immediately, with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Third solution. The following proof, which frequently appears in texts on probability and statistics where (2.5) is especially useful, is independent of the methods used up to this point.

Let $A = \frac{1}{n}\sum_{i=1}^{n} a_i$ and $Q = \sqrt{\frac{1}{n}\sum_{i=1}^{n} a_i^2}$. Then

$$0 \le \sum_{i=1}^{n} (a_i - A)^2$$

$$= \sum_{i=1}^{n} (a_i^2 - 2Aa_i + A^2)$$

$$= nQ^2 - 2A \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} A^2$$

$$= nQ^2 - 2An\left(\frac{1}{n} \sum_{i=1}^{n} a_i\right) + nA^2$$

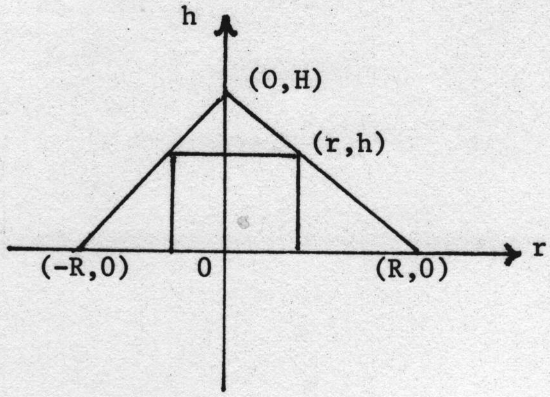
$$= n(Q^2 - 2A^2 + A^2) = n(Q^2 - A^2),$$

whence $A^2 \le Q^2$ with equality if and only if $a_i = A$ for every i, or, what is the same thing, if and only if $a_1 = a_2 = \cdots = a_n$.

2.3. The maximum will occur at the simultaneous maximum of $(s - a)(s - b)(s - c)(s - d)$ and minimum of $abcd \cos^2 \frac{\varepsilon}{2}$, if such exists. The first occurs at $s - a = s - b = s - c = s - d$, or $a = b = c = d$, while the second occurs at $\frac{\varepsilon}{2} = \frac{\pi}{2}$. Both occur only in the case of a square.

Remark. The following more general theorem can be proved easily using geometric methods: the n-sided polygon of greatest area with given perimeter is the regular polygon.

2.4. Let V be the volume of the cylinder, r the radius of its base, and h its altitude. Then $V = \pi r^2 h$. From the figure and the two-intercept form of a line, we have that r and h are related by $\frac{r}{R} + \frac{h}{H} = 1$.



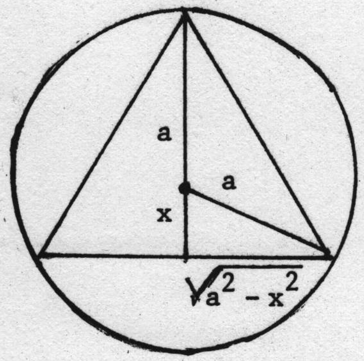
Thus we may write

$$V = 4\pi R^2 H\left(\frac{r}{2R}\right)\left(\frac{r}{2R}\right)\left(\frac{h}{H}\right)$$

$$\le 4\pi R^2 H \left(\frac{\frac{r}{2R} + \frac{r}{2R} + \frac{h}{H}}{3}\right)^3 = \frac{4}{27}\pi R^2 H ,$$

with the maximum of V occurring when $\frac{r}{2R} = \frac{h}{H} = \frac{1}{3}$, or $r = \frac{2R}{3}$, $h = \frac{H}{3}$. Thus the volume of the cylinder is at most 4/9 the volume of the cone.

2.5. (a) In the figure, the altitude is $a + x$ and the radius of the base is $\sqrt{a^2 - x^2}$. It is easy to see that x must be positive for the cone of maximum volume, but this is irrelevant to our proof. The volume of the cone is

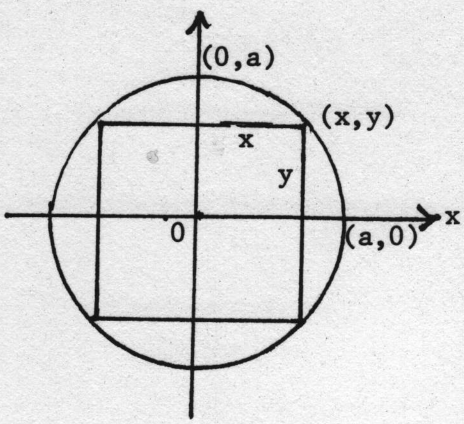


$$V = \frac{\pi}{3}(a^2 - x^2)(a + x)$$

$$= \frac{4\pi}{3}\left(\frac{a + x}{2}\right)\left(\frac{a + x}{2}\right)(a - x).$$

Since the sum of the factors is constant, 2a, the maximum occurs when $\frac{a + x}{2} = a - x$, or $x = a/3$. This determines the cone.

(b) Let V be the volume of the cylinder. Then, from the figure,



$$V = 2\pi x^2 y$$

and

$$x^2 + y^2 = a^2.$$

Thus

$$V^2 = 16\pi^2\left(\frac{x^2}{2}\right)\left(\frac{x^2}{2}\right)(y^2) \le 16\pi^2\left(\frac{a^2}{3}\right)^3,$$

with equality when $\frac{x^2}{2} = y^2 = \frac{a^2}{3}$, or $x = a\sqrt{\frac{2}{3}}$, $y = \frac{a}{\sqrt{3}}$.

Remark. We could have also written $V = 2\pi x^2\sqrt{a^2 - x^2}$ or $V = 2\pi y(a^2 - y^2)$. In either case, we square and proceed as before. As in the preceding problem, we see that we can solve this problem without squaring when we use the latter form, $V = 2\pi y(a - y)(a + y)$.

2.6. The maximum of $(a + x)^3(a - x)^4$ occurs at the maximum of $(\frac{a + x}{3})^3(\frac{a - x}{4})^4$, which is $(\frac{2a}{7})^7$, by (2.2), occurring where $\frac{a + x}{3} = \frac{a - x}{4}$, or $x = -a/7$.

Note. This technique of introducing the proper coefficients to make a constant sum or other quantity is widely applicable. It is generalized in the weighted forms of inequalities, as in Section 5.2.

2.7. Substitute a_1^3 for a_1, b_1^3 for b_1, and so on. We are then required to show that

$$(a_1^3 + b_1^3)(a_2^3 + b_2^3)(a_3^3 + b_3^3) \ge (a_1a_2a_3 + b_1b_2b_3)^3 ,$$

or, after expansion and cancellation,

$$(a_1^3a_2^3b_3^3 + a_2^3a_3^3b_1^3 + a_3^3a_1^3b_2^3) + (b_1^3b_2^3a_3^3 + b_2^3b_3^3a_1^3 + b_3^3b_1^3a_2^3)$$

$$\ge 3a_1^2a_2^2a_3^2b_1b_2b_3 + 3b_1^2b_2^2b_3^2a_1a_2a_3 .$$

It suffices to show that the first quantity in parentheses on

the left side is at least the first term on the right and similarly for the second quantities. But these follow directly from (2.2). Equality holds if and only if $a_1a_2b_3 = a_2a_3b_1 = a_3a_1b_2$ and $b_1b_2a_3 = b_2b_3a_1 = b_3b_1a_2$. Dividing these sets of equalities as written, and letting $x_i = a_i/b_i$, $i = 1,2,3$, we get

$$\frac{x_1x_2}{x_3} = \frac{x_2x_3}{x_1} = \frac{x_3x_1}{x_2},$$

or

$$x_1 = x_2 = x_3 .$$

that is, $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$ is implied by the sets of equalities. Furthermore, this set also implies the original two. Thus, they are equivalent, and our inequality becomes an equality if and only if the a's and the b's are proportional.

<u>Note</u>. This inequality is a special case of Hölder's inequality (4.15).

2.8. From (2.2), $\sqrt[10]{a_1a_2a_2a_3a_3a_3a_4a_4a_4a_4} \le \frac{a_1+a_2+a_2+a_3+a_3+a_3+a_4+a_4+a_4+a_4}{10}$, from which the desired inequality follows immediately. Equality holds if and only if $a_1 = a_2 = a_3 = a_4$.

<u>Remark</u>. In this inequality, we have the exponents in the geometric mean going over to coefficients in the arithmetic mean. This will be generalized and expanded in Chapter V.

2.9. (a) We have $(1 - x)^5(1 + x)(1 + 2x)^2 \le \left(\frac{5(1-x) + (1+x) + 2(1+2x)}{8}\right)^8 = 1$, with equality if and only if $1 - x = 1 + x = 1 + 2x$, or $x = 0$. (2.2) is applicable only if all the factors are positive. If

they are not, exactly two of them must be negative for a maximum to occur. These must be $(1 - x)^5$ and $(1 + x)$. But these cannot both be negative. Hence, the solution is complete.

<u>Remark</u>. See the remark following the solution to Problem 2.8. See also Problem 2.46.

(b) Note that $x + 5 = 5x - 7 = 11 - x$ has the common solution $x = 3$, where each factor equals 8. Now $x + 1\Big|_{x=3} = 4 = \frac{1}{2} \cdot 8$. Thus, we have

$$\frac{1}{2}(x+5)^2(5x-7)(11-x)^9(2x+2) \le$$

$$\le \frac{1}{2}\left(\frac{2(x+5) + (5x-7) + 9(11-x) + (2x+2)}{13}\right)^{13}$$

$$= \frac{1}{2}\left(\frac{104}{13}\right)^{13} = 2^{38}.$$

The maximum is thus 2^{38} at $x = 3$. On $x > 0$, only one factor may be negative. Hence, (2.2) is applicable for finding the maximum, and the solution is complete.

2.10. By (2.2) for $n = 3$ and $n = 2$

$$\frac{a+b+c}{3} = \frac{(a+b)+(b+c)(c+a)}{3} \cdot \frac{1}{2} \ge \frac{1}{2}\sqrt[3]{(a+b)(b+c)(c+a)}$$

$$= \sqrt[3]{\left(\frac{a+b}{2}\right)\left(\frac{b+c}{2}\right)\left(\frac{c+a}{2}\right)} \ge \sqrt[3]{\sqrt{ab}\sqrt{bc}\sqrt{ca}} = \sqrt[3]{abc}\ .$$

In each case, equality holds if and only if $a = b = c$.

2.11. The $\binom{n}{m}$ terms of σ_n contain each x_i exactly $\binom{n-1}{m-1}$ times. Hence, by (2.2),

$$\frac{\sigma_m}{\binom{n}{m}} \geq \left[\prod_{i=1}^{n} x_i^{\binom{n-1}{m-1}}\right]^{\frac{1}{\binom{n}{m}}} = \prod_{i=1}^{n} x_i^{m/n} = \sigma_n^{m/n},$$

since $\binom{n}{m} = \dfrac{n(n-1)!}{m(m-1)!(n-m)!} = \dfrac{n}{m} \cdot \binom{n-1}{m-1}$. Equality holds if and only if $x_1 = x_2 = \cdots = x_n$. Now by expansion and definition of σ_i,

$$\prod_{i=1}^{n}(1 + x_i) = 1 + \sum_{i=1}^{n}\sigma_i \geq 1 + \sum_{m=1}^{n}\binom{n}{m}\sigma_n^{m/n}$$

$$= \sum_{m=0}^{n}\binom{n}{m}(\sqrt[n]{\sigma_n})^m 1^{n-m} = (1 + \sqrt[n]{\sigma_n})^n$$

by the binomial theorem. Equality holds if and only if $x_1 = \cdots = x_n$.

2.12. (a) First solution. Let the two numbers be x and y. Assume that $x < y$. If a is such that $1 < a < \frac{y}{x}$, then ax and $\frac{1}{a}y$ are closer than x and y in the sense that $|ax - \frac{y}{a}| < y - x$. Thus, we are required to show that

$$ax + \frac{1}{a}y < x + y.$$

From the conditions on a, we obtain $x < \frac{1}{a}y$, or $(a - 1)x < \frac{a-1}{a}y = (1 - \frac{1}{a})y$, or $ax + \frac{1}{a}y < x + y$, as desired.

Second solution. Let $xy = rs$ be such that r and s are closer than x and y and let $x < y$. Then $x < r$, and, if we multiply this inequality by the positive quantity $y - r$, we obtain $xy - xr < yr - r^2$, or $xy + r^2 < (x + y)r$, or $x + y > r + \frac{xy}{r} = r + s$, as desired.

(b) If not all the numbers in (2.2) are equal, then there must be two of them, say a_1 and a_2, such that $a_1 < G < a_2$, where $G = (\prod_{i=1}^{n} a_i)^{1/n}$. If we let $a_1' = G$ and $a_2' = a_1 a_2/G$, then a_1' and a_2' are closer than a_1 and a_2 are, and therefore, by part (a), $a_1' + a_2' < a_1 + a_2$, or

$$\frac{a_1' + a_2' + a_3 + \cdots + a_n}{n} < \frac{a_1 + a_2 + a_3 + \cdots + a_n}{n} .$$

Repeating the process at most n-1 times, we arrive at a set of numbers $a_1'' = a_2'' = \cdots = a_n'' = G$ such that

$$\frac{1}{n} \sum_1^n a_i > \frac{1}{n} \sum_1^n a_i'' = G = \left(\prod_1^n a_i\right)^{1/n} .$$

Thus, (2.2) is true and equality holds if and only if $a_1 = \cdots = a_n$.

2.13. It will be shown that (2.2) is true for $n = 2^r$, where r is any integer, and that if (2.2) is true for $n = k$, then it is true for $n = k - 1$. We shall then be able to conclude that (2.2) is true for all n.

We have seen in (2.1) that (2.2) is true for $n = 2^1$, so that we may assume that (2.2) is true for $n = 2^r$, or that

$$(1) \qquad \frac{1}{2^r} \sum_{i=1}^{2^r} a_i \geq \left(\prod_{i=1}^{2^r} a_i\right)^{\frac{1}{2^r}} ,$$

where $a_1, \cdots, a_{2^r}$ are any positive numbers, and where equality holds if and only if $a_1 = a_2 = \cdots = a_{2^r}$. Consider any set of $2^{r+1} = 2 \cdot 2^r$ numbers, $a_1, \cdots, a_{2^r}, a_{2^r+1}, \cdots, a_{2^{r+1}}$. By (1) and (2.1), we have

$$\frac{1}{2^{r+1}}\sum_{i=1}^{2^{r+1}} a_i = \frac{\left(\frac{1}{2^r}\sum_{i=1}^{2^r} a_i\right) + \left(\frac{1}{2^r}\sum_{i=2^r+1}^{2^{r+1}} a_i\right)}{2}$$

$$\geq \frac{\left(\prod_1^{2^r} a_i\right)^{\frac{1}{2^r}} + \left(\prod_{2^r+1}^{2^{r+1}} a_i\right)^{\frac{1}{2^r}}}{2}$$

$$\geq \left[\left(\prod_1^{2^r} a_i\right)^{\frac{1}{2^r}} \left(\prod_{2^r+1}^{2^{r+1}} a_i\right)^{\frac{1}{2^r}}\right]^{\frac{1}{2}}$$

$$= \left(\prod_{i=1}^{2^{r+1}} a_i\right)^{\frac{1}{2^{r+1}}},$$

or

$$\frac{1}{2^{r+1}}\sum_{i=1}^{2^{r+1}} a_i \geq \left(\prod_{i=1}^{2^{r+1}} a_i\right)^{\frac{1}{2^{r+1}}},$$

with equality holding if and only if $a_1 = a_2 = \cdots = a_{2^r}$, $a_{2^r+1} = a_{2^r+2} = \cdots = a_{2^{r+1}}$, and $a_1 a_2 \cdots a_{2^r} = a_{2^r+1} a_{2^r+2} \cdots a_{2^{r+1}}$, that is, if and only if $a_1 = a_2 = \cdots = a_{2^{r+1}}$. Thus by the principle of mathematical induction, (2.2) holds for all $n = 2^r$, where r is any integer. That is, (2.2) holds for arbitrarily large r. Now assume that (2.2) is true for n = k:

$$\frac{1}{k}\sum_{i=1}^{k} a_i \geq \left(\prod_{i=1}^{k} a_i\right)^{1/k}, \tag{2}$$

for any $a\ ,a_2,\cdots,a_k$. Let $a_k = \frac{1}{k-1}\sum_{i=1}^{k-1} a_i$. Then by (2)

$$a_k = \frac{1}{k}((k-1)a_k + a_k) = \frac{1}{k}\sum_{i=1}^{k} a_i \geq \left(\prod_{i=1}^{k} a_i\right)^{1/k},$$

or

$$a_k^k \geq \prod_{i=1}^{k} a_i = \left(\prod_1^{k-1} a_i\right) a_k.$$

Thus $a_k^{k-1} \geq \prod_1^{k-1} a_i$, or

$$\frac{1}{k-1} \sum_{i=1}^{k-1} a_i \geq \left(\prod_{i=1}^{k-1} a_i\right)^{\frac{1}{k-1}},$$

where equality holds if and only if $a_1 = a_2 = \cdots = a_{k-1} = a_k \equiv$ $\equiv \frac{1}{k-1} \sum_1^{k-1} a_i$, which is to say, $a_1 = a_2 = \cdots = a_{k-1}$. Thus (2.2) holds for $n = k - 1$ if it holds for $n = k$. Therefore, by the principle of backward induction, (2.2) holds for all n.

2.14. The only main steps that need be added are those that prove the equality condition. Along with assuming inductively that $A_k \geq G_k$, we assume that $A_k = G_k$ implies $a_1 = \cdots = a_k$. Then $A_{k+1} = G_{k+1}$ implies that $A_k = G_k$ and $A = G$ (also, that $A_k = A$). This implies that $a_1 = \cdots = a_k$ and $a_{k+1} = A_{k+1}$, or $a_{k+1} = a_1$. Therefore, $a_1 = \cdots = a_{k+1}$ is a necessary condition for equality. It is also clearly sufficient.

2.15. For $n = 1$, (2.2) is clearly true. Now assume (2.2) for $n = k$. If we use the notation given, A_k, a mean of $a_1, \cdots, a_k$, is not greater than a_{k+1}, which is at least each of $a_1, \cdots, a_k$. Thus $b \geq 0$. Furthermore, if $b = 0$, then $a_1 + a_2 = \cdots = a_{k+1}$. Now

$$A_{k+1} = \frac{\sum_1^k a_i + a_{k+1}}{k+1} = \frac{kA_k + A_k + b}{k+1} = A_k + \frac{b}{k+1}.$$

Thus

$$\begin{aligned}(A_{k+1})^{k+1} &= (A_k + \frac{b}{k+1})^{k+1} \\ &= A_k^{k+1} + (k+1)(A_k)^k(\frac{b}{k+1}) + \frac{(k+1)k}{2!}(A_k)^{k-1}(\frac{b}{k+1})^2 + \\ &\qquad + \cdots + (\frac{b}{k+1})^{k+1} \\ &\geq A_k^{k+1} + (k+1)(A_k)^k(\frac{b}{k+1}) \\ &= A_k^{k+1} + bA_k^k \\ &= A_k^k\, a_{k+1} \geq (a_1a_2\cdots a_k)a_{k+1}\ ,\end{aligned}$$

where we have used the binomial theorem. Equality holds if and only if $b = 0$. Thus (2.2) is true for $n = k + 1$. By induction, then, (2.2) is true for all n and equality holds if and only if $b = 0$, that is, $a_1 = a_2 = \cdots = a_n$.

<u>Remark</u>. We have used, and proved, the inequality $(1 + h)^n \geq 1 + nh$, where $h \geq 0$ and n is a positive integer, with equality holding if and only if $h = 0$ or $n = 1$. In fact, this inequality, known as <u>Bernoulli's inequality</u>, is true for any real $n \geq 1$. If $0 \leq n \leq 1$, the sign of the inequality is reversed. See Problem 4. .

2.16. For $n = 1$, (2.2) is vacuously satisfied. Assume (2.2) for $n = k$. If $x \geq 1$, then $x^k \geq x^{k-1}$, $x^k \geq x^{k-2}$, $\cdots$, $x^k \geq 1$, with equality in each case if and only if $x = 1$. Adding, we get

$$kx^k \geq (x^{k-1} + x^{k-2} + \cdots + 1),$$

or

$$kx^k(x - 1) \geq (x^{k-1} + \cdots + 1)(x - 1) = x^k - 1.$$

Rearranged, this is

$$(1)\qquad (k+1)x^k - 1 \le kx^{k+1},$$

with equality if and only if $x = 1$. Similarly, if $0 < x \le 1$, $x^k \le x^{k-1}, \cdots, x^k \le 1$, which gives $kx^k \le (x^{k-1} + \cdots + 1)$, or

$$kx^k(x-1) \ge x^k - 1.$$

Again, we get (1). Substituting $x = \sqrt[k]{c_1 \cdots c_k}$, where $c_1, \cdots, c_k$ are to be determined (they depend on the $a_1, \cdots, a_k$ of (2.2)), we have

$$(k+1)c_1 \cdots c_k - 1 \le k(c_1^{k+1} \cdots c_k^{k+1})^{1/k} \le c_1^{k+1} + \cdots + c_k^{k+1},$$

by the inductive hypothesis. Letting $c_i = b_i/b_{k+1}$, where $b_1, \cdots, b_k$ are also to be determined, we have

$$b_1^{k+1} + b_2^{k+1} + \cdots + b_{k+1}^{k+1} \ge (k+1)b_1 b_2 \cdots b_{k+1},$$

after adding 1 to both sides and multiplying through by b_{k+1}^{k+1}. Thus, letting $b_i^{k+1} = a_i$ gives (2.2). Equality holds if and only if $x = 1$ and $c_1 = \cdots = c_k$, or $\frac{b_1}{b_{k+1}} = \cdots = \frac{b_k}{b_{k+1}} = 1$, or $a_1 = \cdots = a_k$. Thus, by induction, (2.2) is proved.

2.17. By (2.2) we have

$$(n!)^{1/n} = (1 \cdot 2 \cdots n)^{1/n} < \frac{1 + 2 + \cdots + n}{n}$$

$$= \frac{1}{n} \cdot \frac{n}{2}(n+1) = \frac{n+1}{2},$$

whence

$$n! < \left(\frac{n+1}{2}\right)^n.$$

Equality cannot hold for any value of $n > 1$ because no two of the factors in $n!$ are equal.

2.18. (a) This problem is simply the dual of Problem 1.30. Thus, the box which minimizes the surface area in this problem is the same as that which maximizes the volume in Problem 1.30, namely, the cube.

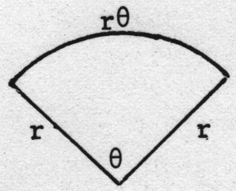
(b) First solution. Reflect the box in the plane of the missing face. The original box and this reflection form a box possessing all six faces and whose surface area and volume are twice those of the original. The volume of this new box is maximized when it is a cube. This maximum occurs at the maximum of the original box, which must therefore be half a cube, whose biggest face is the one opposite the missing face.

Second solution. Let a,b be the edges of the missing face and c the other edge. Let S be the surface area and V the volume. Then $S = ab + 2ac + 2bc$ and $V = abc$, so that

$$V^2 = \frac{1}{4}(ab)(2ac)(2bc) \le \frac{1}{4}\left(\frac{S}{3}\right)^3 ,$$

with equality at $ab = 2ac = 2bc$, or $a = b = 2c$. This is half a cube, as in the first solution.

2.19. Let r be the radius of the circle and θ the angle of the sector. If P is the perimeter and A the area of the sector, then $P = 2r + r\theta$ and $A = r^2\theta/2 =$



$= \frac{1}{4}(2r)(r\theta) \le \frac{1}{4}(P/2)^2$. Thus, the maximum area is $P^2/16$, where $\theta = 2$ radians and $r = P/4$.

2.20. Using (2.2) with $n = 3$ and (2.4), we have

$$4\sqrt{3}\sqrt{(x + y + z)(xyz)} \le 4\sqrt{3}\sqrt{(x + y + z)\left(\frac{x + y + z}{3}\right)^3} = \frac{4}{3}(x + y + z)^2$$

$$= \frac{1}{3}[(x + y) + (y + z) + (z + x)]^2 \le (x + y)^2 + (y + z)^2 + (z + x)^2.$$

Equality holds if and only if $x = y = z$.

2.21. We begin with the inequality

$$(y + z)(z + x)(x + y) \ge 8xyz,$$

which is proved easily by using (2.1) (see Problem 1.). Equality holds if and only if $x = y = z$. Let us substitute $x = s - a$, $y = s - b$, $z = s - c$ (note that $x,y,z > 0$). We obtain $abc \ge 8(s - a)(s - b)(s - c)$, or $sabc \ge 8s(s - a)(s - b)(s - c) = 8K^2$, whence

$$\frac{abc}{4K} \ge \frac{2K}{s}, \quad \text{or} \quad 2r \le R,$$

with equality if and only if $a = b = c$, the equilateral triangle.

Remark. In geometry, it is shown that the square of the distance between the centers of the two circles is $R(R - 2r)$, and hence that $R - 2r \ge 0$. The centers coincide only for the equilateral triangle, of course.

2.22. From (1.2), we have that

$$\sqrt{a_1^2 + (1 - a_2)^2} \ge \frac{\sqrt{2}}{2}(|a_1| + |1 - a_2|) \ge \frac{\sqrt{2}}{2}(a_1 + 1 - a_2)$$

$$\sqrt{a_2^2 + (1 - a_3)^2} \ge \frac{\sqrt{2}}{2}(|a_2| + |1 - a_3|) \ge \frac{\sqrt{2}}{2}(a_2 + 1 - a_3)$$

$$\cdots$$

$$\sqrt{a_n^2 + (1 - a_1)^2} \ge \frac{\sqrt{2}}{2}(|a_n| + |1 - a_1|) \ge \frac{\sqrt{2}}{2}(a_n + 1 - a_1),$$

with equality in the first inequalities of each line if and only if $|a_1| = |1 - a_2|$, $|a_2| = |1 - a_3|$, $\cdots$, $|a_n| = |1 - a_1|$, and in the second if and only if $a_1 \geq 0$, $1 - a_2 \geq 0$, $a_2 \geq 0$, $1 - a_3 \geq 0$, $\cdots$, $a_n \geq 0$, $1 - a_1 \geq 0$; i.e., $0 \leq a_i \leq 1$ for $i = 1,\cdots,n$. Adding, we obtain (where $a_{n+1} = a_1$)

$$\sum_{i=1}^{n} \sqrt{a_i^2 + (1 - a_{i+1})^2} \geq n \frac{\sqrt{2}}{2} ,$$

with equality if and only if both equality conditions above are satisfied. That is, if n is even,

$$a_1 = 1 - a_2 = a_3 = 1 - a_4 = \cdots = a_{n-1} = 1 - a_n = a_1,$$

or

$$a_1 = a_3 = \cdots = a_{n-1} = a; \quad a_2 = a_4 = \cdots = a_n = 1 - a,$$

for any a such that $0 \leq a \leq 1$. If n is odd,

$$a_1 = 1 - a_2 = a_3 = 1 - a_4 = \cdots = 1 - a_{n-1} = a_n = 1 - a_1,$$

or

$$a_1 = a_2 = \cdots = a_n = \frac{1}{2} .$$

2.23. (a) This problem was solved in Problem 1. . We have, by (2.1), that $A \leq (P/4)^2$, where A and P are the area and perimeter of the rectangle.

(b) Since each person has a width, the maximum number of people to sit around the table determines the perimeter of three of its sides. If we reflect the table in the plane of the wall, we get a new table with twice the perimeter and twice

the area of the original. This new table, however, has all four sides included in its perimeter. It has maximum area when it is a square. Hence, the original table has maximum area when it is half a square, the longer side being against the wall.

Note. Compare with Problem 2.18.

2.24. Using (2.6), we have

$$\begin{aligned}(1 + \tfrac{1}{x})(1 + \tfrac{1}{y})(1 + \tfrac{1}{z}) &= 1 + (\tfrac{1}{x} + \tfrac{1}{y} + \tfrac{1}{z}) + (\tfrac{1}{xy} + \tfrac{1}{yz} + \tfrac{1}{zx}) + \tfrac{1}{xyz} \\ &\ge 1 + (\frac{9}{x + y + z}) + \frac{3}{(xyz)^{2/3}} + (\frac{3}{x + y + z})^3 \\ &\ge 1 + \frac{9}{x + y + z} + \frac{3}{(\frac{x + y + z}{3})^2} + (\frac{3}{x + y + z})^3 \\ &= 1 + 9 + 27 + 27 = 64.\end{aligned}$$

Equality holds if and only if $x = y = z = 1/3$.

Remark. We cannot use the inequalities $1 + \frac{1}{x} \ge 2\sqrt{\frac{1}{x}}$, $1 + \frac{1}{y} \ge 2\sqrt{\frac{1}{y}}$, $1 + \frac{1}{z} \ge 2\sqrt{\frac{1}{z}}$, for equality holds in all cases if and only if $x = y = z = 1$, which contradicts the hypothesis that $x + y + z = 1$. In other words, equality would never hold.

2.25. First solution. By (2.6),

$$\begin{aligned}y = \frac{1}{1 - x} + \frac{1}{1 + x} + \frac{2}{1 + x^2} &= \frac{2}{1 - x^2} + \frac{2}{1 + x^2} \\ &\ge \frac{2^2}{\left(\frac{1 - x^2 + 1 + x^2}{2}\right)} = 4\end{aligned}$$

with equality if and only if $x = 0$. The requirement that $|x| < 1$ guarantees that $1 - x^2$ is positive.

<u>Second solution</u>. Since

$$y = \frac{2}{1 - x^2} + \frac{2}{1 + x^2} = \frac{4}{1 - x^4},$$

clearly y is minimized at $x = 0$, where $y = 4$.

2.26. By the lemma of Section 2.2,

$$(x + a)(y - a) > xy,$$

where $0 < x < y$ and $0 < a < y - x$. This gives

$$\frac{x + y}{xy} > \frac{x + y}{(x + a)(y - a)},$$

or

$$\frac{1}{x} + \frac{1}{y} > \frac{1}{x + a} + \frac{1}{y - a}, \quad \text{or} \quad \frac{2}{\frac{1}{x + a} + \frac{1}{y - a}} > \frac{2}{\frac{1}{x} + \frac{1}{y}},$$

which is what is required.

2.27. On the interval specified, all terms are positive. Hence, by (2.6),

$$y = \frac{1}{1 - x} + \frac{1}{1 - x} + \frac{1}{1 - x} + \frac{1}{1 - x} + \frac{1}{1 - x} + \frac{1}{1 + x} + \frac{1}{1 + 2x} + \frac{1}{1 + 2x}$$

$$\geq \frac{8^2}{5(1 - x) + (1 + x) + 2(1 + 2x)} = 8,$$

with equality if and only if $1 - x = 1 + x = 1 + 2x$, or $x = 0$.

2.28. Let V be the volume of the cone, H its altitude, and R the radius of its base. Then, as in Problem 2.4,

$$\frac{r}{R} + \frac{h}{H} = 1$$

and

$$V = \frac{1}{3}\pi R^2 H = \frac{1}{12}\pi r^2 h\left(\frac{2R}{r}\right)\left(\frac{2R}{r}\right)\left(\frac{H}{h}\right)$$

$$\geq \frac{1}{12}\pi r^2 h\left(\frac{3}{\frac{r}{2R} + \frac{r}{2R} + \frac{h}{H}}\right)^3 = \frac{9}{4}\pi r^2 h\ ,$$

with equality when $\frac{2R}{r} = \frac{H}{h} = 3$, or $R = \frac{3r}{2}$, $H = 3h$. The volume of the cone is thus at least 9/4 that of the cylinder.

2.29. <u>First solution</u>. Squaring, we have

$$y^2 = (x^2)^m(a^2 - x^2)^{2n} = \frac{1}{\alpha^m}(\alpha x^2)^m(a^2 - x^2)^{2n},$$

where we want $m\alpha - 2n = 0$, or $\alpha = 2n/m$. Thus,

$$y^2 \leq \frac{1}{\alpha^m}\left(\frac{m\alpha x^2 + 2n(a^2 - x^2)}{m + 2n}\right)^{m+2n} = \left[\left(\frac{m}{2n}\right)\left(\frac{a^2}{\frac{m}{2n} + 1}\right)^{1+\frac{2n}{m}}\right]^m ,$$

with equality at $\frac{2n}{m}x^2 = a^2 - x^2$, or $x = \frac{a}{\sqrt{1 + \frac{2n}{m}}}$.

<u>Second solution</u>. Without squaring, we have

$$y = \frac{1}{\alpha^m}\frac{1}{\beta^n}(\alpha x)^m[\beta(a - x)]^n(a + x)^n$$

We wish to have $m\alpha - n\beta + n = 0$ and $x = \frac{a}{\alpha - 1} = \frac{\beta - 1}{\beta + 1}a$, or $\frac{1}{\alpha - 1} = \frac{m\alpha}{m\alpha + 2n}$, so that

$$\alpha = 1 + \sqrt{1 + \frac{2n}{m}}\ .$$

This shows that y is maximized at $x = \frac{a}{\sqrt{1 + \frac{2n}{m}}}$.

2.30. With three undetermined coefficients, α, β, γ, we wish to maximize

$$(\alpha x)[\beta(a - x)][\gamma(a + x)](2a - x).$$

We have

(1) $$\alpha - \beta + \gamma - 1 = 0$$

and

$$x = \frac{2}{\alpha + 1} a = \frac{2 - \gamma}{\gamma + 1} a = \frac{\beta - 2}{\beta - 1} a ,$$

or

(2) $$2(\gamma + 1) = (\alpha + 1)(2 - \gamma)$$

and

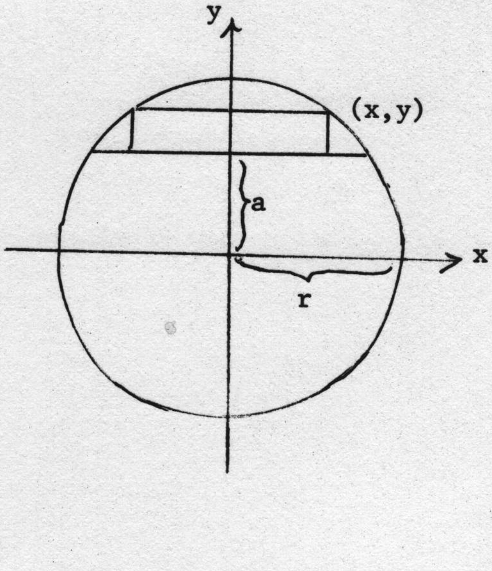
(3) $$2(\beta - 1) = (\alpha + 1)(\beta - 2).$$

If we eliminate α and β from these three equations, we obtain

$$\gamma^3 - 8\gamma^2 + 8\gamma - 1 = 0.$$

One root of this equation is $\gamma = 1$, which gives $\alpha = \beta = 3$, so that $x = \frac{a}{2}$ is a point at which the product is greatest. The other roots are $\gamma = (7 \pm 3\sqrt{5})/2$, both of which are positive. However, from (2), $\alpha = 3\gamma/(2 - \gamma)$, and, since $\alpha > 0$, $\gamma < 2$. Thus, $\gamma = (7 - 3\sqrt{5})/2$. But then $\alpha = \sqrt{5} - 2 > 0$ and $\beta = (1 - \sqrt{5})/2 < 0$. Thus, the only possible coefficients are $\alpha = \beta = 3$, $\gamma = 1$.

2.31. (a) Let the radius of the circle be r and the distance from the chord to the center be a, as in the figure. The point (x,y) on the circle is one corner of the rectangle. If A is the area of the rectangle, we have



$x^2 + y^2 = r^2$ and $A = 2x(y - a) = 2\sqrt{r^2 - y^2}(y - a)$, or

$$A^2 = 4(r^2 - y^2)(y - a)^2 = 4(r - y)(r + y)(y - a)(y - a).$$

Introducing two undetermined coefficients, α and β, we have

$$A^2 = \frac{4}{\alpha\beta}[\alpha(r - y)][\beta(r + y)](y - a)(y - a),$$

and we want

$$(1) \qquad -\alpha + \beta + 2 = 0$$

and a common solution to

$$\begin{cases} \alpha(r - y) = y - a \\ \alpha(r - y) = \beta(r + y). \end{cases}$$

The solution of the first of these is $y = (\alpha r + a)/(1 + \alpha)$, and a solution of the second is

$$(2) \qquad y = \frac{\alpha - \beta}{\alpha + \beta} r = \frac{r}{\alpha - 1},$$

where we have eliminated β from (1). Thus, $(\alpha r + a)/(1 + \alpha) = r/(\alpha - 1)$, or

$$\alpha = \frac{2r - a + \sqrt{a^2 + 8r^2}}{2r},$$

since α must be positive. Also, $\beta = \alpha - 2$ is positive, since $r > a$, $a^2 + 8r^2 > a^2 + 4ra + 4r^2 = (a + 2r)^2$, or

$\beta = \dfrac{-a - 2r + \sqrt{a^2 + 8r^2}}{2r} > 0$. From (2), we have, therefore, that $y = \frac{1}{4}(a + \sqrt{a^2 + 8r^2})$. Since $r > y > a$, this determines the rectangle. We remark that when $a = 0$, we get half a square, as we should.

(b) Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the chord $y = d$. If (x,y), $x > 0$, is a corner of the rectangle on the ellipse, then

$$A = 2x(y - d) = 2ab\left(\frac{x}{a}\right)\left(\frac{y}{b} - \frac{d}{b}\right),$$

where we have

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

From part (a), the maximum of A occurs at

$$\frac{y}{b} = \frac{\frac{d}{b} + \sqrt{\left(\frac{d}{b}\right)^2 + 8}}{4},$$

or

$$y = \frac{d + \sqrt{d^2 + 8b^2}}{4}.$$

2.32. (a) Let the sphere be $x^2 + y^2 + z^2 = r^2$ and let the plane bounding the segment be $z = a > 0$. If V is the volume of the box and (x,y,z) a corner of the box on the sphere in the first octant, then $V = (2x)(2y)(z - a) \le 2(x^2 + y^2)(z - a) =$ $= 2(r - z)(r + z)(z - a)$, with equality if and only if $x = y$. We introduce the undetermined coefficients α and β:

$$V = \frac{2}{\alpha\beta}[\alpha(r - z)][\beta(r + z)](z - a),$$

where $-\alpha + \beta + 1 = 0$, and, as in Problem 2.31(a),

$$z = \frac{\alpha r + a}{1 + \alpha} = \frac{\alpha - \beta}{\alpha + \beta}r = \frac{r}{2\alpha - 1}, \text{ or } \alpha = \frac{r - a + \sqrt{a^2 + 3r^2}}{2r}.$$

Since $r > a$, $\beta = \alpha - 1$ is positive. Therefore, the maximum occurs at $z = \frac{1}{3}(a + \sqrt{a^2 + 3r^2})$.

(b) The method of solution is analogous to that used in Problem 2.31(b). Let the ellipsoid be $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and the bounding plane be $z = d > 0$. If V is the volume and (x,y,z) the corner of the box on the ellipsoid with $x,y > 0$, then

$$V = 4xy(z - d) = 4abc(\frac{x}{a})(\frac{y}{b})(\frac{z}{c} - \frac{d}{c}) ,$$

and, since $(\frac{x}{a})^2 + (\frac{y}{b})^2 + (\frac{z}{c})^2 = 1$, the maximum occurs, by part (a), at $x/a = y/b$, and

$$\frac{z}{c} = \frac{\frac{d}{c} + \sqrt{(\frac{d}{c})^2 + 3}}{3}, \quad \text{or} \quad z = \frac{1}{3}(d + \sqrt{d^2 + 3c^2}).$$

2.33. (a) Let the radius of the cylinder be r, the height z, and the surface area S. Then $z + r^2 = c$ and $S = 2\pi r^2 + 2\pi rz =$ $= 2\pi r(r + z) = 2\pi r(c + r - r^2) = 2\pi r(r_1 - r)(r - r_2)$, where $r_1r_2 = -c$, $r_1 + r_2 = 1$, and $r_1 > r_2$. We introduce undetermined coefficients, α and β, with

$$S = \frac{2\pi}{\alpha\beta} r(\alpha r_1 - \alpha r)(\beta r - \beta r_2).$$

We require that $1 - \alpha + \beta = 0$ and

$$x = \frac{\alpha r_1}{1 + \alpha} = \frac{\beta r_2}{\beta - 1},$$

so that

$$\frac{(\beta + 1)r_1}{\beta + 2} = \frac{\beta r_2}{\beta - 1},$$

or

$$\beta = \frac{r_2 + \sqrt{r_1^2 + r_2^2 - r_1r_2}}{r_1 - r_2}.$$

Then β and $\alpha = \beta + 1$ are positive, so that our solution is, with $\Delta = \sqrt{r_1^2 + r_2^2 - r_1 r_2}$,

$$x = \frac{\beta r_2}{\beta - 1} = \frac{r_2(r_2 + \Delta)}{2r_2 - r_1 + \Delta} = \frac{r_2(r_2 + \Delta)(2r_2 - r_1 - \Delta)}{4r_2^2 - 4r_1r_2 + r_1^2 - r_2^2 - r_1^2 + r_1r_2}$$

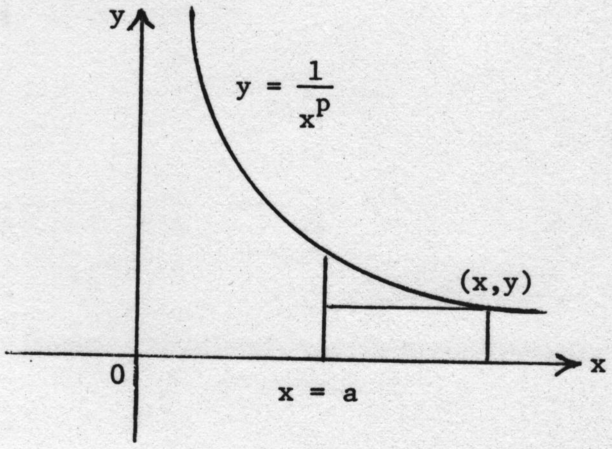
$$= \frac{r_2(2r_2^2 - r_1r_2 + (r_2 - r_1)\Delta - r_1^2 - r_2^2 + r_1r_2)}{3r_2^2 - 3r_1r_2}$$

$$= \frac{r_2^2 - r_1^2 + (r_1 - r_1)\Delta}{3(r_2 - r_1)} = \frac{r_1 + r_2 + \Delta}{3} = \frac{1 + \Delta}{3}$$

$$= \frac{1 + \sqrt{1 + 3c}}{3} ;$$

the last step follows from the fact that $\Delta = \sqrt{(r_1 + r_2)^2 - 3r_1r_2}$. This is valid only if $x \leq c$, which holds if and only if $c \geq 1$.

(b) Let $(x,y,z), x,y,z \geq 0$, be one corner of the box and let S be its surface area. Then $S = 8xy + 4xz + 4yz = 8xy + 4(x + y)z$. For fixed z, this is maximized, as in Problem 2.32(a), when $y = x$; we then want to maximize $S = 8x^2 + 8xz$ with $z + x^2 = c$. Thus $S = 8x(c + x - x^2)$. By the solution to part (a), this is maximized at $x = \frac{1}{3}(1 + \sqrt{1 + 3c})$. Note that the box is inscribed in the cylinder of part (a).



2.34. If A is the area of the rectangle, then

$$A = (x - a)y = \frac{1}{x^p}(x - a)$$

$$= \frac{1}{x^{p-1}}(1 - \frac{a}{x})$$

$$= (\frac{p-1}{a})^{p-1}(\frac{a}{(p-1)x})^{p-1}(1 - \frac{a}{x})$$

$$\le (\frac{p-1}{a})^{p-1}\left[\frac{(p-1)\frac{a}{(p-1)x} + (1 - \frac{a}{x})}{p}\right]^p$$

$$= (\frac{p-1}{a})^{p-1}\frac{1}{p^p},$$

with equality if and only if $\frac{a}{(p-1)x} = 1 - \frac{a}{x}$, or $x = pa/(p - 1)$.

2.35. Let the surface area be S, the volume V, the altitude h, and one side of the base s. To specify the shape, let the area of the base be a and its perimeter p when s = 1. Then $V = has^2$ and

$$S = 2as^2 + hps = 2as^2 + \frac{1}{2}hps + \frac{1}{2}hps$$

$$\ge 3(\frac{1}{2}ap^2h^2s^4)^{1/3} = 3(\frac{1}{2}\frac{p^2}{a}V^2)^{1/3},$$

which is given, and where equality holds if and only if $2as^2 = \frac{1}{2}hps = (\frac{1}{2}\frac{p^2}{a}V^2)^{1/3}$ or $s = \sqrt[3]{\frac{pV}{4a^2}}$, $h = \sqrt[3]{\frac{16aV}{p^2}}$. (This also gives $h/s = 4a/p$.)

2.36. Let K be the area of the triangle, let α be the given angle, let a be the opposite side, and let b and c be the including sides. We have $K = \frac{1}{2}bc \sin \alpha$.

(a) By (2.1), $b + c \ge 2\sqrt{bc} = 2\sqrt{\frac{2K}{\sin\alpha}}$, with equality if and only if $b = c$.

(b) By (2.1) and the law of cosines,

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

$$\geq 2bc(1 - \cos \alpha) = \frac{4K(1 - \cos \alpha)}{\sin \alpha}$$

with equality if and only if $b = c$.

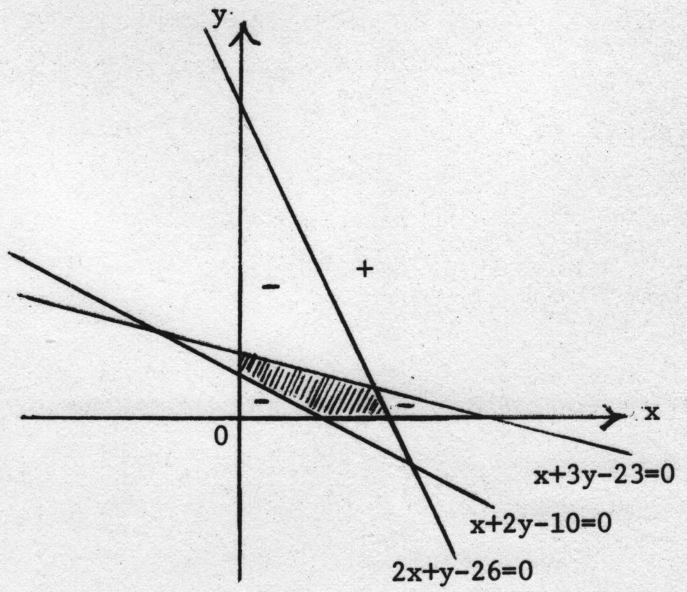
(c) Since a and $b + c$ are both minimized when the triangle is isosceles, so is their sum, the perimeter.

2.37. On the domain given, at most one factor in z is negative. Thus, by (2.2),

$$z \leq \left(\frac{x + 2y + 11 - 3x - y + 2x - y + 1}{3}\right)^3 = 64$$

with equality if and only if $x + 2y = 11 - 3x - y = 2x - y + 1$, or $x = 2$, $y = 1$.

2.38. In the graph on the right, the signs indicate the sign of z in that domain. z is also positive in the shaded region. On the other positive region, all three factors in z as written are positive and hence there is no relative maximum there,



since there is no absolute maximum of z. All relative maxima in the first octant are therefore in the shaded region, where z can be written as

$$z = (x + 2y - 10)(23 - x - 3y)(26 - 2x - y),$$

with all factors positive.

To use (2.2), we introduce two undetermined coefficients, α and β:

$$z = \frac{1}{\alpha\beta}[\alpha(x + 2y - 10)][\beta(23 - x - 3y)](26 - 2x - y),$$

where we want the sum of the factors to be independent of x and y, or $\alpha - 2 - \beta = 0$ and $2\alpha - 1 - 3\beta = 0$, so that $\alpha = 5$, $\beta = 3$. Thus

$$z \leq \frac{1}{15}\left(\frac{-50 + 69 + 26}{3}\right)^3 = 225,$$

with equality if and only if $5(x + 2y - 10) = 3(23 - x - 3y) = 26 - 2x - y$, or $x = 3$, $y = 5$, which is in the shaded region.

2.39. Let the numbers be a and b. Then, by (2.1),

$$\frac{1}{2}\left(\frac{a + b}{2} + \frac{2}{\frac{1}{a} + \frac{1}{b}}\right) = \frac{1}{2}\left(\frac{a + b}{2} + \frac{2ab}{a + b}\right) \geq \sqrt{ab},$$

with equality if and only if the arithmetic and harmonic means are equal, that is, if and only if $a = b$.

2.40. <u>First solution</u>. From (2.6), we have that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9 \quad \text{or} \quad bc + ca + ab \geq 9abc,$$

or

$$1 - (a + b + c) + (bc + ca + ab) - abc \geq 8abc,$$

which, when factored, yields

$$(1 - a)(1 - b)(1 - c) \geq 8abc,$$

or

$$\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right) \geq 8,$$

with equality if and only if $a = b = c = \frac{1}{3}$.

Second solution. We desire to show that

$$(1 - a)(1 - b)(1 - c) \geq 8abc \ .$$

Since $1 - a = b + c$, etc., this follows from Problem 1.5.
Equality holds if and only if $a = b = c = \frac{1}{3}$.

2.41. By (1.5) and (2.6),

$$(a^2 + b^2 + c^2)(\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab}) \geq (bc + ca + ab)(\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab}) \geq 9,$$

or $(a + b + c)(a^2 + b^2 + c^2) \geq 9abc$. Also by (2.6), we have $(a + b + c)(bc + ca + ab) \geq 9abc$. Adding these two inequalities gives us

$$(a + b + c)(a^2 + b^2 + c^2 + bc + ca + ab) \geq 18abc,$$

or

$$(a + b + c)(2a^2 + 2b^2 + 2c^2) \geq (a+b+c)(a^2+b^2+c^2 - ab - bc - ca) + 18abc$$

$$= a^3 + b^3 + c^3 = 3abc + 18abc,$$

or

$$a^3 + b^3 + c^3 + 15abc \leq 2(a + b + c)(a^2 + b^2 + c^2),$$

with equality if and only if $a = b = c$.

2.42. First solution. By (2.6),

$$\left(\frac{1}{ab^2} + \frac{1}{bc^2} + \frac{1}{ca^2}\right)(ab^2 + bc^2 + ca^2) \geq 9$$

or $$(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 9a^2b^2c^2,$$

with equality if and only if $a = b = c$.

<u>Second solution</u>. By (2.2), $a^2b + b^2c + c^2a \geq 3abc$ and $ab^2 + bc^2 + ca^2 \geq 3abc$, both with equality if and only if $a = b = c$. Multiplication gives the desired inequality.

2.43. The minimum of the distance d from (4,-2,1) to any point, (x,y,z), on the plane is the distance we seek.

By (2.5), we have

$$d = \sqrt{(x-4)^2 + (y+2)^2 + (z-1)^2} = \sqrt{4\left(\frac{4-x}{2}\right)^2 + 9\left(\frac{y+2}{3}\right)^2 + 36\left(\frac{1-z}{6}\right)^2}$$

$$\geq \frac{4\left(\frac{4-x}{2}\right) + 9\left(\frac{y+2}{3}\right) + 36\left(\frac{1-z}{6}\right)}{\sqrt{4+9+36}} = \frac{-2x+3y-6z+13+7}{7} = 1,$$

with equality at $\frac{4-x}{2} = \frac{y+2}{3} = \frac{1-z}{6} = \frac{1}{7}$, or $x = 26/7$, $y = -\frac{11}{7}$, $z = \frac{1}{7}$. Thus, $(\frac{26}{7}, -\frac{11}{7}, \frac{1}{7})$ is the point on the plane closest to (4, -2, 1); its distance from the plane is 1.

<u>Remark</u>. For this problem, it would be simpler to substitute $x = 4$, $y = -2$, $z = 1$ into the normal form of the equation of the plane:

$$\frac{2x - 3y + 6z - 13}{\sqrt{2^2 + (-3)^2 + 6^2}} = 0$$

to obtain $d = \left|\frac{2(4) - 3(-2) + 6(1) - 13}{7}\right| = 1.$

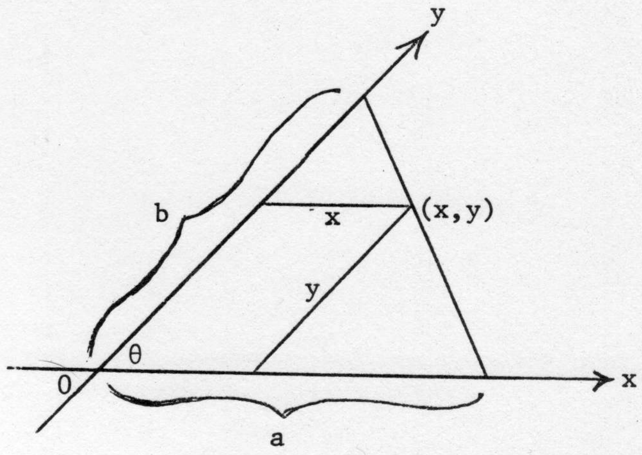
2.44. From (1.1), we have

$$\sum_1^n \sqrt{(x_i - A)^2 + (B - y_i)^2} \geq \sum_1^n \frac{\sqrt{2}}{2}(x_i - A + B - y_i)$$

$$= n\,\frac{\sqrt{2}}{2}(B - A) + \frac{\sqrt{2}}{2}(\Sigma x_i - \Sigma y_i) = n\,\frac{\sqrt{2}}{2}(B - A),$$

with equality if and only if $x_i - A = B - y_i$ $(i = 1,\cdots,n)$, or $x_i + y_i = A + B$.

2.45. Along the rays of the angle, θ, we set up oblique coordinate axes, as in the figure. If (x,y) is the given point, then, by the two-intercept form of a straight line,



$$\frac{x}{a} + \frac{y}{b} = 1.$$

The area of the triangle, K, is given by $\frac{1}{2}ab \sin \theta$, and thus, by (2.6),

$$K = \frac{1}{2}\,ab \sin \theta = \frac{xy \sin \theta}{2}\left(\frac{a}{x}\right)\left(\frac{b}{y}\right)$$

$$\geq \frac{xy \sin \theta}{2}\left(\frac{2}{\frac{x}{a} + \frac{y}{b}}\right)^2 = 2xy \sin \theta,$$

with equality if and only if $\frac{a}{x} = \frac{b}{y} = 2$, or $a = 2x$, $b = 2y$. The point (x,y) is then the mid-point of that side of the triangle.

SOLUTIONS FOR CHAPTER III

3.1. If we set $2x + 1 = y$ or $x = \frac{1}{2}(y - 1)$, the expression $(3x + 2)(x + 1)/(2x + 1)$ becomes

$$\frac{(\frac{3}{2}(y - 1) + 2)(\frac{1}{2}(y - 1) + 1)}{y} = \frac{3y^2 + 4y + 1}{4y}$$

$$= \frac{1}{4}\{3y + \frac{1}{y} + 4\} = \frac{1}{4}\{(\sqrt{3y} - \frac{1}{\sqrt{y}})^2 + 4 + 2\sqrt{3}\},$$

whose minimum value $\frac{1}{4}(4 + 2\sqrt{3})$ is achieved when $(\sqrt{3y} - \frac{1}{\sqrt{y}})^2 = 0$, or when $y = 1/\sqrt{3}$, or $x = \frac{1}{2}(\frac{\sqrt{3}}{3} - 1)$.

3.2. We write $\frac{7}{x^2 + 4 + 5}$ as $\frac{7}{(x + 2)^2 + 1}$, and remark that the largest value of the function occurs when the denominator is a minimum. The denominator is a minimum when $(x + 2)^2 = 0$ or $x = -2$, and the value of the maximum of the given function is 7.

3.3. Let us set $y = x + 3$ or $x = y - 3$, so that the original expression becomes

$$\frac{y}{y^2 - 2y + 2} = \frac{1}{y + \frac{2}{y} - 2} = \frac{1}{(\sqrt{y} - \sqrt{\frac{2}{y}})^2 + 2\sqrt{2} - 2}.$$

The denominator of this last expression is a minimum when $(\sqrt{y} - \sqrt{\frac{2}{y}})^2 = 0$, or $y = \sqrt{2}$, or $x = \sqrt{2} - 3$; the maximum value of the given expression is then $\frac{1}{2\sqrt{2} - 2}$.

3.4. Again, the maximum of the function occurs when we minimize the denominator, and, in particular, when we minimize $\cos^4 x + \sin^4 x$.

By (1.9), we know that the minimum of $\cos^4 x + \sin^4 x$ is 1/2, and that this minimum is achieved when $x = \pi/4$. Hence the maximum value of the given function is 6/11, which is achieved when $x = \pi/4$.

3.5. If we apply (1.3) of Chapter I to part of the denominator, we have

$$x^2 + y^2 + \frac{1}{x^2} + \frac{1}{y^2} \ge \frac{1}{2}(x + y)^2 + \frac{1}{2}(\frac{1}{x} + \frac{1}{y})^2$$
$$= \frac{1}{2} + \frac{1}{2}(\frac{1}{xy})^2 ,$$

with equality only if $x = y$. On the other hand, $2\sqrt{xy} \le x + y = 1$, or $1/(xy) \ge 4$, with equality under the same conditions, namely that $x = y$. Hence

$$\frac{1}{2} + \frac{1}{2}(\frac{1}{xy})^2 \ge 17/2,$$

so that

$$\frac{7}{x^2 + y^2 + 7 + \frac{1}{x^2} + \frac{1}{y^2}} \le \frac{7}{\frac{17}{2} + 7} = \frac{14}{31} .$$

Thus the maximum is 14/31 and is achieved when $x = y = 1/2$.

3.6. For the first factor of the denominator we have

$$2 + (x + y)(\frac{1}{x} + \frac{1}{y}) = 2 + 2 + \frac{y}{x} + \frac{x}{y} \ge 2 + 2 + 2 = 6 ,$$

with equality when $y/x = 1$, or $x = y$. For the second factor we have

$$1 + \sqrt{x + y} + \frac{1}{\sqrt{x + y}} \ge 1 + 2 = 3 ,$$

with equality when $\sqrt{x + y} = 1$ or $x + y = 1$. The two equality conditions will be compatible if $x = y = 1/2$, and the maximum is $18/18 = 1$.

3.7. The identity given in the hint is easy to show, and we omit the details. If we denote first by S the quantity tan A + tan B + tan C, we have from (2.2) that

$$S \geq 3\sqrt[3]{\tan A \tan B \tan C} = 3\sqrt[3]{S}\,,$$

or $S^3 \geq 27S$, or $S \geq 3\sqrt{3}$, with equality if and only if tan A = = tan B = tan C.

It now follows from (2.4) that

$$\tan^2 A + \tan^2 B + \tan^2 C \geq \frac{1}{3}S^2 \geq \frac{1}{3}(3\sqrt{3})^2\,,$$

with equality again if and only if tan A = tan B = tan C, so that the minimum value of the given expression is 9 and is achieved only in the case of the equilateral triangle.

Remark. A general approach to problems of this kind is given in Chapter V.

3.8. Suppose that the equation of $\mathcal{L}$ is $ax + by + c = 0$. If $\mathcal{L}$ is then written in normal form, $x \cos \omega + y \sin \omega - p = 0$, where $\cos \omega = a/\sqrt{a^2 + b^2}$, $\sin \omega = b/\sqrt{a^2 + b^2}$, then, when (x,y) is replaced by (x_o,y_o), the expression $x_o \cos \omega + y_o \sin \omega - p$ is the (signed) distance from the point (x_o,y_o) to the line $\mathcal{L}$. In particular, the distance d_i of the point (x_i,y_i) to $\mathcal{L}$ is simply $x_i \cos \omega + y_i \sin \omega - p$. We remark first that $\bar{x} \cos \omega + \bar{y} \sin \omega - p = 0$ since $(\bar{x},\bar{y})$ lies on $\mathcal{L}$. If M is the total mass $\sum_{i=1}^{n} m_i$ of the system, then the total first moment of the system about $\mathcal{L}$ is

$$\sum_{i=1}^{n} d_i m_i = \sum_{i=1}^{n} m_i(x_i \cos\omega + y_i \sin\omega - p)$$

$$= \left(\sum_{i=1}^{n} m_i x_i\right)\cos\omega + \left(\sum_{i=1}^{n} m_i y_i\right)\sin\omega - p\sum_{i=1}^{n} m_i$$

$$= (\bar{x}M)\cos\omega + (\bar{y}M)\sin\omega - pM$$

$$= M(\bar{x}\cos\omega + \bar{y}\sin\omega - p) = 0.$$

3.9. Let the coordinates of A,B,C,D be (x_1,y_1,z_1), (x_2,y_2,z_2), (x_3,y_3,z_3), (x_4,y_4,z_4), respectively. Then, if the coordinates of P are (x,y,z), we have

$$PA^2 + PB^2 + PC^2 + PD^2 = \sum_{i=1}^{4}\{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2\}$$

$$= 4x^2 - 2x\sum_{i=1}^{4} x_i + \sum_{i=1}^{4} x_i^2$$

$$+ 4y^2 - 2y\sum_{i=1}^{4} y_i + \sum_{i=1}^{4} y_i^2 + 4z^2 - 2z\sum_{i=1}^{4} z_i + \sum_{i=1}^{4} z_i^2 .$$

We have here the sum of three independent quadratic expressions of the form (3.1), and if we complete the squares, we see that the minimum occurs at the point

$$x = \frac{x_1 + x_2 + x_3 + x_4}{4}, \quad y = \frac{y_1 + y_2 + y_3 + y_4}{4}, \quad z = \frac{z_1 + z_2 + z_3 + z_4}{4} .$$

3.10. The definitions and notation are given in the statement of the problem. Let $ax + by + cz + d = 0$ be the equation of the plane, which we transform to its normal form $x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0$ by dividing by $\sqrt{a^2 + b^2 + c^2}$. We remark first that $(\bar{x},\bar{y},\bar{z})$ must lie on this plane and satisfy the equation

$\overline{x}\cos\alpha + \overline{y}\cos\beta + \overline{z}\cos\gamma - p = 0$. As in Problem 3.8, the distance d_i of (x_i, y_i, z_i) to Π is given by $x_i \cos\alpha + y_i \cos\beta + z_i \cos\gamma - p$, so that the total first moment is

$$\begin{aligned}\sum_{i=1}^{n} m_i d_i &= \sum_{i=1}^{n} m_i(x_i \cos\alpha + y_i \cos\beta + z_i \cos\gamma - p)\\ &= \left(\sum_{i=1}^{n} m_i x_i\right)\cos\alpha + \left(\sum_{i=1}^{n} m_i y_i\right)\cos\beta + \left(\sum_{i=1}^{n} m_i z_i\right)\cos\gamma - p\left(\sum_{i=1}^{n} m_i\right)\\ &= M\overline{x}\cos\alpha + M\overline{y}\cos\beta + M\overline{z}\cos\gamma - Mp\\ &= M(\overline{x}\cos\alpha + \overline{y}\cos\beta + \overline{z}\cos\gamma - p) = 0.\end{aligned}$$

3.11. The total second moment, or moment of inertia, is the sum

$$I_{\mathcal{L}} = PA^2 + PB^2 + PC^2 + PD^2 = PB^2 + PC^2 + PD^2,$$

since $PA = 0$ and $m_j = 1$ $(j = 1,2,3,4)$. Since $\mathcal{L}$ passes through the center of gravity, or centroid, of triangle BCD, $PB^2 + PC^2 + PD^2$ takes its minimum value, namely,

$C_1 + C_2 - \frac{1}{3}(x_1 + x_2 + x_3)^2 - \frac{1}{3}(y_1 + y_2 + y_3)^2$, where C_1 and C_2 are the quantities given in Example 4.

3.12. First solution. $9 = (1 + 1 + 1)^2 = \left(\frac{1}{\sqrt{x}}\sqrt{x} + \frac{1}{\sqrt{y}}\sqrt{y} + \frac{1}{\sqrt{z}}\sqrt{z}\right)^2$ $\le \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)(x + y + z)$ with equality for $x = y = z$.

Second solution. In problems of this sort, that is, where we replace 1 by $\frac{1}{\alpha}\cdot\alpha$, the relation between the arithmetic and geometric means is usually applicable. Thus, we have

$$x + y + z \ge 3\sqrt[3]{xyz} \quad \text{and} \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \ge 3\sqrt[3]{\frac{1}{xyz}}\ .$$

If we multiply these two inequalities, we obtain

$$(x + y + z)(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}) \geq 9 \sqrt[3]{xyz}\,\frac{1}{\sqrt[3]{xyz}} = 9 ,$$

with equality for $x = y = z$ since this is the condition for equality in <u>both</u> inequalities involving arithmetic and geometric means.

3.13. The idea is the same as in the solution of Problem 3.12. For the first method of solution we have

$$n^2 = (1 + 1 + \cdots + 1)^2 = \left(\frac{1}{\sqrt{x_1}} \sqrt{x_1} + \cdots + \frac{1}{\sqrt{x_n}} \sqrt{x_n}\right)^2$$

$$\leq \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)(x_1 + \cdots + x_n) ,$$

with equality for $x_1 = \cdots = x_n$. Using arithmetic and geometric means, we have

$$x_1 + \cdots + x_n \geq n\sqrt[n]{x_1 \cdots x_n}$$

$$\frac{1}{x_1} + \cdots + \frac{1}{x_n} \geq n\sqrt[n]{\frac{1}{x_1 \cdots x_n}} ,$$

whence $$(x_1 + \cdots + x_n)\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right) \geq n^2 ,$$

with equality for $x_1 = \cdots = x_n$.

3.14. If we write $1 - \cos x = 2 \sin^2 \frac{x}{2}$ and $1 + \cos x = 2 \cos^2 \frac{x}{2}$, we have

$$\frac{A}{1 - \cos x} + \frac{B}{1 + \cos x} = \frac{A}{2} \csc^2 \frac{x}{2} + \frac{B}{2} \sec^2 \frac{x}{2} ,$$

which is essentially the form of the function which is treated in Example 8. Thus, by the Cauchy-Schwarz inequality,

$$\frac{A}{2}\csc^2\frac{x}{2} + \frac{B}{2}\sec^2\frac{x}{2} = \left(\frac{A}{2}\csc^2\frac{x}{2} + \frac{B}{2}\sec^2\frac{x}{2}\right)\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right)$$

$$\geq \left(\sqrt{\frac{A}{2}}\csc\frac{x}{2}\sin\frac{x}{2} + \sqrt{\frac{B}{2}}\sec\frac{x}{2}\cos\frac{x}{2}\right)^2 = \frac{1}{2}(\sqrt{A} + \sqrt{B})^2 .$$

The minimum occurs when equality prevails, namely, whenever

$$\frac{\sqrt{\frac{A}{2}}\csc\frac{x}{2}}{\sin\frac{x}{2}} = \frac{\sqrt{\frac{B}{2}}\sec\frac{x}{2}}{\cos\frac{x}{2}}, \text{ or whenever } x = 2\arctan\sqrt[4]{\frac{A}{B}} .$$

3.15. By the Cauchy-Schwarz inequality, we have

$$3x + 4\sqrt{1 - x^2} \leq (3^2 + 4^2)^{1/2}\,(x^2 + (1 - x^2))^{1/2} = 5.$$

The maximum occurs whenever equality prevails, which happens whenever $\frac{3}{x} = \frac{4}{\sqrt{1 - x^2}}$, or whenever $x = 3/5$. The form of the function suggests a <u>second solution</u>: If we set $x = \cos\theta$, then $\sin\theta = \sqrt{1 - x^2}$, so that the function to be maximized is $3\cos\theta + 4\sin\theta = 5\cos(\theta - \theta_o)$, where $\theta_o = \arctan\frac{4}{3}$. The maximum value of $5\cos(\theta - \theta_o)$ is 5 and occurs when $\cos(\theta - \theta_o) = 1$ or when $\theta = \theta_o = \arctan 4/3$, which is the same as the first solution.

3.16. By the Cauchy-Schwarz inequality we have that

$$(12x + 3y + 4z)^2 \leq (12^2 + 3^2 + 4^2)(x^2 + y^2 + z^2) = 13^2,$$

so that $-13 \leq 12x + 3y + 4z \leq 13$. The extreme values occur when $x/12 = y/12 = z/4$. The largest value, 13, occurs when $x = 12/13$, $y = 3/13$, $z = 4/13$, while the smallest value, -13, occurs when $x = -12/13$, $y = -3/13$, $z = -4/13$.

3.17. It is enough to compute the minimum value of $\sec^6 x + \csc^6 x$ and then multiply by 3. By Problem 1.10, we have
$\sec^6 x + \csc^6 x \ge \frac{1}{2^2}(\sec^2 x + \csc^2 x)^3$ with equality for $\sec x = \csc x$, or $x = \pi/4$. Now, by the Cauchy-Schwarz inequality, we have

$$\frac{1}{2^2}(\sec^2 x + \csc^2 x)^3 = \frac{1}{4}[(\sec^2 x + \csc^2 x)(\cos^2 x + \sin^2 x)]^3$$

$$\ge \frac{1}{4}[(\sec x \cos x + \csc x \sin x)^2]^3 = \frac{1}{4}[(2)^2]^3 = 2^4 = 16,$$

with equality for $\cos x = \sin x$, or $x = \pi/4$, <u>the same value as in the first inequality</u>. Hence the minimum value of $3 \sec^6 x + 3 \csc^6 x$ is $3 \cdot 16 = 48$, and occurs when $x = \pi/4$.

3.18. Using the Cauchy-Schwarz inequality, we have

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = \frac{1}{\sqrt{2}}\frac{x}{\sqrt{2}} + \frac{1}{\sqrt{3}}\frac{y}{\sqrt{3}} + \frac{1}{\sqrt{6}}\frac{z}{\sqrt{6}} \le \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right)^{1/2}\left(\frac{x^2}{2} + \frac{y^2}{3} + \frac{z^2}{6}\right)^{1/2}$$

$$= \sqrt{\frac{x^2}{2} + \frac{y^2}{3} + \frac{z^2}{6}},$$

with equality for $x = y = z$.

3.19. We may write the given function as

$$\left(\frac{x + y + z}{y + z} - 1\right) + \left(\frac{x + y + z}{z + x} - 1\right) + \left(\frac{x + y + z}{x + y} - 1\right)$$

$$= (x + y + z)\left(\frac{1}{y + z} + \frac{1}{z + x} + \frac{1}{x + y}\right) - 3$$

$$= \frac{1}{2}[(x + y) + (y + z) + (z + y)]\left(\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x}\right) - 3.$$

By the result of Problem 3.12, this last expression is not less than $\frac{1}{2} \cdot 3^2 - 3$, so that

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \ge \frac{1}{2} \cdot 3^2 - 3 = 3/2.$$

Equality holds if and only if $x = y = z$.

3.20. We have $AP + PB = \sqrt{x^2 + y^2 + a^2} + \sqrt{(1-x)^2 + (1-y)^2 + b^2}$, which, by (3.20), is not less than

$$\sqrt{(x + (1-x))^2 + (y + (1-y))^2 + (a+b)^2} = \sqrt{2 + (a+b)^2}\ .$$

The minimum of $AP + PB$ occurs when we have equality in (3.20), or when $\frac{x}{1-x} = \frac{y}{1-y} = \frac{a}{b}$, or when $x = y = \frac{a}{a+b}$. The geometrical proof is almost identical to the geometrical proof in Example 7 if we observe that the perpendicular lines dropped from A and B to the (x,y)-plane form a plane.

3.21. We have that $(xy + yz + zx)^2 \le (x^2 + y^2 + z^2)(y^2 + z^2 + x^2)$, whence

$$|xy + yz + zx| \le x^2 + y^2 + z^2\ .$$

Equality occurs whenever $x = y = z$.

3.22. Let us apply the Cauchy-Schwarz inequality to the expression

$$\{[(a_1b_1)(c_1d_1) + (a_2b_2)(c_2d_2) + (a_3b_3)(c_3d_3) + (a_4b_4)(c_4d_4)]^2\}^2$$
$$\le \{[(a_1b_1)^2 + (a_2b_2)^2 + (a_3b_3)^2][(c_1d_1)^2 + (c_2d_2)^2 + (c_3d_3)^2]\}^2$$
$$= (a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2)^2(c_1^2d_1^2 + c_2^2d_2^2 + c_3^2d_3^2)^2$$
$$\le (a_1^4 + a_2^4 + a_3^4)(b_1^4 + b_2^4 + b_3^4)(c_1^4 + c_2^4 + c_3^4)(d_1^4 + d_2^4 + d_3^4).$$

3.23. By (3.25) we have

$$x = \int_0^x \sec t \cos t \, dt < \left(\int_0^x \sec^2 t \, dt\right)^{1/2} \left(\int_0^x \cos^2 t \, dt\right)^{1/2}$$

$$= \sqrt{\tan x} \left(\int_0^x \left(\frac{1}{2} + \frac{1}{2}\cos 2t\right)dt\right)^{1/2} = \sqrt{\tan x}\, \left(\tfrac{1}{4}(2x + \sin 2x)\right)^{1/2},$$

whence $4x^2 < \tan x\,(2x + \sin 2x)$, from which the desired inequality follows.

3.24. If we write $x_i = 1 \cdot x_i$, we have

$$(x_1 + x_2 + \cdots + x_n)^2 = (1 \cdot x_1 + 1 \cdot x_2 + \cdots + 1 \cdot x_n)^2$$

$$\le (1^2 + 1^2 + \cdots + 1^2)(x_1^2 + \cdots + x_n^2) = n(x_1^2 + \cdots + x_n^2) \ ,$$

with equality if $x_1 = \cdots = x_n$.

3.25. If we apply the Cauchy-Schwarz inequality to the right-hand side of

$$\left(\sum_{i=1}^n a_i x_i\right)^2 = \left(\sum_{i=1}^n \sqrt{a_i}\, (\sqrt{a_i}\, x_i)\right)^2,$$

we have

$$\left(\sum_{i=1}^n a_i x_i\right)^2 \le \left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n a_i x_i^2\right) = \sum_{i=1}^n a_i x_i^2 \, .$$

as was desired. Equality holds if and only if $\sqrt{a_i}\, x_i/\sqrt{a_i} = \sqrt{a_k}\, x_k/\sqrt{a_k}$ or $x_i = x_k$ for all i and k, or $x_1 = \cdots = x_n$. In Problem 3.18 $a_1 = 1/2$, $a_2 = 1/3$, $a_3 = 1/6$ so that $a_1 + a_2 + a_3 = 1$, and in Problem 3.24 $a_1 = \cdots = a_n = 1/n$. A general theory of inequalities of this sort will be given in Chapter V.

3.26. If we write $a_1x_1 + a_2x_2 + \cdots + a_nx_n = a_1b_1\left(\frac{x_1}{b_1}\right) + \cdots + a_nb_n\left(\frac{x_n}{b_n}\right)$, then, by the Cauchy-Schwarz inequality,

$$|a_1x_1 + \cdots + a_nx_n| \le \left(a_1^2b_1^2 + \cdots + a_n^2b_n^2\right)^{1/2}\left[\left(\frac{x_1}{b_1}\right)^2 + \cdots + \left(\frac{x_n}{b_n}\right)^2\right]^{1/2}$$

$$= \left(a_1^2b_1^2 + \cdots + a_n^2b_n^2\right)^{1/2},$$

with equality whenever $\frac{x_1}{a_1b_1^2} = \frac{x_2}{a_2b_2^2} = \cdots = \frac{x_n}{a_nb_n^2}$. The minimum value, $-(a_1^2b_1^2 + \cdots + a_n^2b_n^2)^{1/2}$, is achieved by changing the signs of $x_1,\cdots,x_n$.

3.27. Consider the quadratic function of t:

$$y = \int_a^b [\sqrt{p(x)}\, f(x)t - \sqrt{p(x)}\, g(x)]^2 \ge 0 ,$$

which is non-negative and can therefore not have two distinct real roots. We have $y = At^2 + Bt + C$, where

$$A = \int_a^b [f(x)]^2 p(x)dx$$

$$B = -2\int_a^b f(x)g(x)p(x)dx$$

$$C = \int_a^b [g(x)]^2\, p(x)dx.$$

The fact that $B^2 - 4AC \le 0$ implies that

$$\left(\int_a^b f(x)g(x)p(x)dx\right)^2 \le \left(\int_a^b [f(x)]^2p(x)dx\right)\left(\int_a^b [g(x)]^2p(x)dx\right) .$$

The quadratic y can vanish for one value of t, say t_o, in which case we have equality. But then, by the Lemma, since $[\sqrt{p(x)}\, f(x)t_o - \sqrt{p(x)}g(x)]^2$ is non-negative, $g(x) = t_o\, f(x)$, so that g(x) is a multiple of f(x).

3.28. We write $0 \le \sum_{j=1}^{n} \sum_{k=1}^{n} (a_j^2 b_k^2 - 2a_j b_j a_k b_k + a_k^2 b_j^2)$. If we sum first with respect to k, remembering that j is kept fixed in this summation, we obtain

$$0 \le \sum_{j=1}^{n} \{a_j^2(b_1^2 + b_2^2 + \cdots + b_n^2) - 2a_j b_j(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n) + + b_j^2(a_1^2 + a_2^2 + \cdots + a_n^2)\} .$$

Now we sum over j, regarding the three terms in parentheses as constants with respect to j, and we obtain

$$0 \le (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) - - 2(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 + + (b_1^2 + b_2^2 + \cdots + b_n^2)(a_1^2 + a_2^2 + \cdots + a_n^2) ,$$

which gives us the inequality. To determine when equality holds, we see that each of the squares in the double sum given in the statement of the problem must vanish, i.e.,

$$a_j b_k - a_k b_j = 0$$

for all j and k. This means that $\frac{a_j}{b_j} = \frac{a_k}{b_k}$ for all j and k, so that

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n} .$$

3.29. We must show that the series $\sum_{k=1}^{\infty}|a_kb_k|$ is convergent. Let S_n denote the partial sum

$$S_n = |a_1b_1| + |a_2b_2| + \cdots + |a_nb_n| .$$

Since S_n is monotone increasing (that is, $S_n \le S_{n+1}$ for all n), it suffices to show that S_n is bounded independently of n. By the Cauchy-Schwarz inequality, we have

$$S_n = |a_1||b_1| + \cdots + |a_n||b_n| \le (|a_1|^2 + \cdots + |a_n|^2)^{\frac{1}{2}}(|b_1|^2 + \cdots + |b_n|^2)^{\frac{1}{2}} .$$

Now the series of positive terms $\sum_{k=1}^{\infty}|a_k|^2 = \sum_{k=1}^{\infty} a_n^2 = A$ is convergent, so that $|a_1|^2 + \cdots + |a_n|^2 \le A$, and similarly $|b_1|^2 + \cdots + |b_n|^2 \le B$, where $B = \sum_{k=1}^{\infty} b_k^2$. Hence $S_n \le A^{1/2}B^{1/2}$, so that $\lim_{n\to\infty} S_n$ exists. Indeed, we have that $\sum_{k=1}^{\infty}|a_kb_k| \le A^{1/2}B^{1/2}$.

3.30. By Schwarz's inequality we have, for any positive M and N, that

$$\begin{aligned} f^2(N) - f^2(M) &= \int_M^N 2f(x)f'(x)dx \\ &\le 2\left(\int_M^N [f(x)]^2dx\right)^{1/2}\left(\int_M^N [f'(x)]^2dx\right)^{1/2} . \end{aligned}$$

We know that there must exist a sequence $\{x_k\}$ with $\lim_{k\to\infty} x_k = \infty$ such that $\lim_{k\to\infty}[f(x_k)]^2 = 0$, and hence $\lim_{k\to\infty} f(x_k) = 0$, for otherwise there must exist a positive number d such that $[f(x)]^2 \ge d$ for all x greater than some fixed x_o, and this would contradict the assumption that $\int_o^\infty [f(x)]^2dx$ is finite. Next, given $\varepsilon > 0$, we know that, for all M and N greater than some fixed M_o,

$$\int_M^N [f(x)]^2dx < \varepsilon , \qquad \int_M^N [f'(x)]^2dx < \varepsilon ,$$

so that, for all $x > M_o$ and $x_k > M_o$, we have

$$\left|[f(x)]^2 - [f(x_k)]^2\right| \leq 2\left|\int_{x_k}^{x}[f(x)]^2dx\right|^{1/2} \cdot \left|\int_{x_k}^{x}[f'(x)]^2dx\right|^{1/2} < 2\varepsilon,$$

which shows that $\lim_{x\to\infty}[f(x)]^2 = 0$, whence $\lim_{x\to\infty} f(x) = 0$.

Solutions to Chapter IV

4.1. By writing $F(x) = [f(x)]^p$ and $G(x) = [g(x)]^q$, or $f(x) = [F(x)]^\alpha$ and $g(x) = [G(x)]^\beta$, where $\alpha = 1/p$ and $\beta = 1/q$, so that $\alpha + \beta = 1$, we may recast (4.22) in the form

$$\int_a^b [F(x)]^\alpha [G(x)]^\beta dx \le \left(\int_a^b F(x)dx\right)^\alpha \left(\int_a^b G(x)dx\right)^\beta .$$

Suppose next that we have three functions f,g,h and three positive constants α,β,γ with $\alpha + \beta + \gamma = 1$. Then $\beta + \gamma = 1 - \alpha$, from which it follows that

$$\frac{\beta}{1 - \alpha} + \frac{\gamma}{1 - \alpha} = 1.$$

If we write

$$\int_a^b f^\alpha g^\beta h^\gamma dx = \int_a^b f^\alpha \left(g^{\frac{\beta}{1-\alpha}} h^{\frac{\gamma}{1-\alpha}}\right)^{1-\alpha} dx, \quad \text{then}$$

$$\int_a^b f^\alpha g^\beta h^\gamma dx \le \left(\int_a^b f\ dx\right)^\alpha \left(\int_a^b g^{\frac{\beta}{1-\alpha}} h^{\frac{\gamma}{1-\alpha}}\ dx\right)^{1-\alpha}$$

$$\le \left(\int_a^b f\ dx\right)^\alpha \left[\left(\int_a^b g\ dx\right)^{\frac{\beta}{1-\alpha}} \left(\int_a^b h\ dx\right)^{\frac{\gamma}{1-\alpha}}\right]^{1-\alpha}$$

$$= \left(\int_a^b f\ dx\right)^\alpha \left(\int_a^b g\ dx\right)^\beta \left(\int_a^b h\ dx\right)^\gamma,$$

as desired.

4.2. In the last inequality, set $p = \frac{1}{\alpha}$, $q = \frac{1}{\beta}$, $r = \frac{1}{\gamma}$, and make the changes $F = f^\alpha$, $G = g^\beta$, and $H = h^\gamma$.

4.3. We apply Hölder's inequality to the identity

$$3 = 1 + 1 + 1 = \frac{1}{\sqrt{x}}\frac{1}{\sqrt[3]{x}}(\sqrt{x}\,\sqrt[3]{x}) + \frac{1}{\sqrt{y}}\frac{1}{\sqrt[3]{y}}(\sqrt{y}\,\sqrt[3]{y}) + \frac{1}{\sqrt{z}}\frac{1}{\sqrt[3]{z}}(\sqrt{z}\,\sqrt[3]{z})$$

$$\le (\frac{1}{x} + \frac{1}{y} + \frac{1}{z})^{1/2}(\frac{1}{x} + \frac{1}{y} + \frac{1}{z})^{1/3}(x^5 + y^5 + z^5)^{1/6},$$

where the indices $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$, $\gamma = \frac{1}{6}$ have been used. If we now raise both sides to the sixth power, we have the stated inequality, with equality if $x = y = z$.

(Note that this problem is analogous to Problems 3.12 and 3.13, and that an alternative solution is possible:

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \ge 3\sqrt[3]{\frac{1}{xyz}},$$

so that

$$(\frac{1}{x} + \frac{1}{y} + \frac{1}{z})^5 \ge 3^5\sqrt[3]{\frac{1}{(xyz)^5}}.$$

Also

$$x^5 + y^5 + z^5 \ge 3\sqrt[3]{(xyz)^5},$$

so that the multiplication of the last two inequalities also yields a solution of the problem by means of the arithmetic-geometric-mean inequality. This should not be surprising, because we have based our proofs of the Hölder inequalities on the inequality between the arithmetic and geometric means.)

4.4. If we write

$$3 = 1 + 1 + 1 = \frac{1}{x^{1/3}}\frac{1}{x^{1/3}}x^{2/3} + \frac{1}{y^{1/3}}\frac{1}{y^{1/3}}y^{2/3} + \frac{1}{z^{1/3}}\frac{1}{z^{1/3}}z^{2/3},$$

we have

$$3 \le (\frac{1}{x} + \frac{1}{y} + \frac{1}{z})^{1/3}(\frac{1}{x} + \frac{1}{y} + \frac{1}{z})^{1/3}(x^2 + y^2 + z^2)^{1/3},$$

where we have used $\alpha = \frac{1}{3}$, $\beta = \frac{1}{3}$, $\gamma = \frac{1}{3}$. Cubing both sides of the last inequality yields the result, which is the same as Problem 3.12; note the obvious alternative solution and the remarks at the end of the solution of Problem 4.3.

4.5. Write $\alpha = m/n$, so that $0 < m/n < 1$ and m and n are positive integers with $m < n$. Since $1 + x \geq 0$, we may apply the inequality between arithmetic and geometric means to the expression $(1 + x)^\alpha$ to obtain

$$\begin{aligned}(1 + x)^\alpha &= (1 + x)^{m/n} = [(1 + x)^m \, 1^{n-m}]^{1/n} \\ &\leq \frac{1}{n}[(1 + x) + (1 + x) + \cdots + (1 + x) + 1 + 1 + \cdots + 1] \\ &= \frac{1}{n}[m(1 + x) + (n - m)1] = \frac{1}{n}(mx + n) = 1 + \frac{m}{n}x = 1 + \alpha x.\end{aligned}$$

Equality is attained only if all the factors used in the above inequality are equal, namely, only if $1 + x = 1$, or $x = 0$.

4.6. Since $\alpha > 1$, we have $\frac{1}{\alpha} < 1$, and Problem 4.5 yields

$$(1 + \alpha x)^{1/\alpha} \leq 1 + \frac{1}{\alpha} \cdot \alpha x = 1 + x,$$

with equality only for $x = 0$. Raise this last inequality to the power α, and we have

$$1 + \alpha x \leq (1 + x)^\alpha.$$

4.7. In the inequality of Problem 4.6, let us substitute y for $1 + x$, so that

$$y^\alpha \geq 1 + \alpha(y - 1) = 1 - \alpha + \alpha y,$$

with equality only for $y = 1$, so that $y^\alpha - \alpha y \geq 1 - \alpha$. If we now multiply this last inequality by A^α, we have

$$(Ay)^{\alpha} - \alpha A^{\alpha-1}(Ay) \geq (1 - \alpha)A^{\alpha}.$$

If we set $a = \alpha A^{\alpha-1}$ and set $z = Ay$, we have

$$z^{\alpha} - az \geq (1 - \alpha)A^{\alpha}$$

with equality only for $z = A$ (i.e., $y = 1$). But $A = (\frac{a}{\alpha})^{\frac{1}{\alpha-1}}$, and the value of the minimum is $(1 - \alpha)(\frac{a}{\alpha})^{\frac{\alpha}{\alpha-1}}$.

4.8. Suppose first that $1 + \alpha x \geq 0$. Choose a positive integer N large enough so that $0 < -\alpha/N < 1$. Then, by Problem 4.5, we have

$$(1 + x)^{-\alpha/N} \leq 1 - \frac{\alpha}{N} x,$$

or
$$(1 + x)^{\alpha/N} \geq \frac{1}{1 - \frac{\alpha}{N} x} = \frac{1 + \frac{\alpha}{N} x}{1 - \frac{\alpha^2}{N^2} x^2} \geq 1 + \frac{\alpha}{N} x,$$

since $1 - \frac{\alpha^2}{n^2} x^2 \leq 1$. If we raise this last inequality to the power N, we have

$$(1 + x)^{\alpha} \geq (1 + \frac{\alpha}{N})^{N} \geq 1 + \frac{N\alpha}{N} x = 1 + \alpha x.$$

We observe that if $1 + \alpha x$ is negative, the inequality in the statement of the problem is trivially valid.

4.9. For $\alpha < 1$ and for any of the rational numbers r_n such that $\lim r_n = \alpha$, we have

$$(1 + x)^{r_n} \leq 1 + r_n x,$$

so that, in the limit,

$$(1 + x)^{\alpha} = \lim_{r_n \to \alpha} (1 + x)^{r_n} \le \lim_{r_n \to \alpha} (1 + r_n x) = 1 + \alpha x,$$

which proves the inequality, except for the conditions of equality. Suppose now that $x \neq 0$ and $0 < \alpha < 1$; we must show that $(1 + x)^{\alpha}$ is strictly less than $1 + \alpha x$. To this end, let r be a rational number such that $\alpha < r < 1$. Since $(1 + x)^{\alpha} = [(1 + x)^{\alpha/r}]^r$ and since $0 < \frac{\alpha}{r} < 1$, it follows from the result of Problem 4.5 that

$$(1 + x)^{\alpha/r} < 1 + \frac{\alpha}{r} x ;$$

so that
$$(1 + x)^{\alpha} < (1 + \frac{\alpha}{r} x)^r .$$

If we apply the result of Problem 4.5 to the expression $(1 + \frac{\alpha}{r} x)^r$, where we now regard $\frac{\alpha}{r} x$ as the quantity x in Problem 4.5, we have

$$(1 + \frac{\alpha}{r} x)^r < 1 + \frac{r \alpha x}{r} = 1 + \alpha x ,$$

and this gives the desired result. The extension to Problems 4.6 and 4.7 is now immediate. (Let us comment on the labor expended here. It is a well-known fact in analysis that when limits are taken, a strict inequality must be replaced by an inequality in which equality is to be considered, e.g., $0 < 1/n$, but $0 \le \lim_{n\to\infty} 1/n = 0$, so that a separate proof of strict inequality is required in those cases when, indeed, a strict inequality is possible.)

4.10. Consider first x^{α} with $1 + y = x$. Then $(1 + y)^{\alpha} \ge 1 + \alpha y$ with equality for $y = 0$ or $x = 1$, so that $x^{\alpha} \ge 1 + \alpha(x - 1)$ or $x^{\alpha} - \alpha x \ge 1 - \alpha$ with equality for $x = 1$. However, we wish to minimize $x^{\alpha} - ax$ rather than $x^{\alpha} - \alpha x$, and, in order to do this,

we make a change of scale by setting $x = Az$. But if we multiply by A^{α}, we have

$$x^{\alpha}A^{\alpha} - \alpha x\, A^{\alpha} = (1 - \alpha)A^{\alpha},$$

or
$$(xA)^{\alpha} - \alpha A^{\alpha-1}(x\, A) \ge (1 - \alpha)A^{\alpha},$$

where we want $\alpha A^{\alpha-1} = a$, or $A = (\frac{a}{\alpha})^{\frac{1}{\alpha-1}}$. Thus the minimum value is $(1 - \alpha)A^{\alpha} = (1 - \alpha)(\frac{a}{\alpha})^{\frac{\alpha}{\alpha-1}}$, and occurs at the value $x = A$.

4.11. We have $1 = \cos^2 x \cdot 1 + \sin^2 x \cdot 1$

$$\le (\cos^6 x + \sin^6 x)^{1/3}(1^{3/2} + 1^{3/2})^{2/3} = 2^{2/3}(\cos^6 x + \sin^6 x)^{1/3},$$

whence $1 \le 4(\cos^6 x + \sin^6 x)$. Equality holds whenever $\frac{(\cos^2 x)^3}{1^{3/2}} = \frac{(\sin^2 x)^3}{1^{3/2}}$ or $x = \pi/4$. An alternative solution may be found by means of Problem 1.10.

4.12. Again, we have $1 = \cos^2 x \cdot 1 + \sin^2 x \cdot 1$

$$\le (\cos^3 x + \sin^3 x)^{2/3}(1^3 + 1^3)^{1/3} = 2^{1/3}(\cos^3 x + \sin^3 x)^{2/3}.$$

whence $1 \le 2 \cdot (\cos^3 x + \sin^3 x)^2$, which is equivalent to the solution. Equality holds whenever

$$\frac{(\cos^2 x)^{3/2}}{1^3} = \frac{(\sin^2 x)^{3/2}}{1^3}\,, \quad \text{or} \quad x = \pi/4\,.$$

4.13. We start with the identity $1 = \cos^2 x + \sin^2 x$, which is not greater than

$$\left[(\cos^2 x)^{\alpha/2} + (\sin^2 x)^{\alpha/2}\right]^{2/\alpha}\left[1^{\frac{\alpha}{\alpha-2}} + 1^{\frac{\alpha}{\alpha-2}}\right]^{\frac{\alpha-2}{\alpha}},$$

whence

$$1 \le (\cos^\alpha x + \sin^\alpha x)^{2/\alpha}(2)^{\frac{\alpha-2}{\alpha}},$$

or $$1 \le (\cos^\alpha x + \sin^\alpha x)^2\ 2^{\alpha-2},$$

whence $2^{1-\frac{\alpha}{2}} \le \cos^\alpha x + \sin^\alpha x$, with equality for

$$\frac{\cos^\alpha x}{1} = \frac{\sin^\alpha x}{1}, \quad \text{or} \quad x = \pi/4.$$

4.14. Let $1/p = \alpha$ and $1/q = \beta$, so that $\alpha + \beta = 1$, and consider the integral $\int_a^b [F(x)]^\alpha [G(x)]^\beta \phi(x)dx = \int_a^b [F(x)\phi(x)]^\alpha [G(x)\phi(x)]^\beta dx$, which is not less than $(\int_a^b F(x)\phi(x)dx)^\alpha (\int_a^b G(x)\phi(x)dx)^\beta$. If we set $f = F^\alpha$, $g = G^\beta$, then $F = f^{1/\alpha} = f^p$ and $G = g^{1/\beta} = g^q$, and we have

$$\int_a^b f(x)g(x)\phi(x)dx \le \left(\int_a^b (f(x))^p \phi(x)dx\right)^{1/p} \left(\int_a^b (g(x))^q \phi(x)dx\right)^{1/q},$$

with equality if $(f(x))^p = (g(x))^q$.

4.15. Set $p = \frac{n}{n-1}$, so that $1/p + 1/n = 1$. By Hölder's inequality, we have

$$ax + b\sqrt[n]{c^n - x^n} \le (a^p + b^p)^{1/p}(x^n + c^n - x^n)^{1/n} = c(a^p + b^p)^{1/p},$$

with equality, and therefore the maximum, occurring when

$$\frac{a^p}{x^n} = \frac{b^p}{c^n - x^n}, \quad \text{or when} \quad x = \frac{c\ a^{\frac{1}{n-1}}}{\left(a^{\frac{n}{n-1}} + b^{\frac{n}{n-1}}\right)^{1/n}}.$$

4.16. Let $p > 1$ be real and let $q = \frac{p}{p-1}$, so that $\frac{1}{p} + \frac{1}{q} = 1$. We consider the function $x^p - (py)x$, whose minimum is

$$(1 - p)y^{\frac{p}{p-1}} = (1 - p)y^q$$

in Problem 4.10. Thus

$$x^p - (py)x \geq (1 - p)y^q,$$

so that $x^p + (p - 1)y^q \geq p\, x\, y$, or

$$\frac{x^p}{p} + \frac{(p - 1)}{p}\, y^q \geq xy,$$

from which (4.2) follows. From Problem 4.10 it follows that the minimum occurs when

$$x = \left(\frac{py}{p}\right)^{\frac{1}{p-1}} \quad \text{or when} \quad x^p = y^{\frac{p}{p-1}}, \quad \text{or} \quad x^p = y^q,$$

which is the condition for equality in (4.2).

4.17. We set $p = \frac{m}{n}$ $(m > n)$, $q = \frac{r}{s}$ $(r > s)$, and $\frac{1}{r} = 1 - \frac{n}{m} - \frac{s}{r}$. Then

$$\frac{x^p}{p} + \frac{y^q}{q} + \frac{z^r}{r} = \frac{n}{m}\, x^p + \frac{s}{r}\, y^q + \left(1 - \frac{n}{m} - \frac{s}{r}\right) z^r$$

$$= \frac{nrx^p + msy^q + (mr - nr - sm)z^r}{mr}.$$

By the inequality between the arithmetic and geometric means, this last expression is not less than

$$\left\{x^{pnr}\; y^{qms}\; z^{r(mr-nr-sm)}\right\}^{1/mr} = x^{p\frac{n}{m}}\; y^{q\frac{s}{r}}\; z^{r(1-\frac{n}{m}-\frac{s}{r})} = xyz,$$

with equality only if $x^p = y^q = z^r$.

4.18. We write Hölder's inequality in the form

$$\sum_{i=1}^{n} A_i^{1/p} B_i^{1/q} \le \left(\sum_{i=1}^{n} A_i\right)^{1/p}\left(\sum_{i=1}^{n} B_i\right)^{1/q}, \quad \text{where } A_i = a_i^p,\ B_i = b_i^q.$$

Then

$$\sum a_i b_i c_i = \sum A_i^{1/p} B_i^{1/q} c_i = \sum (A_i c_i)^{1/p}(B_i c_i)^{1/q}$$

$$\le (\sum A_i c_i)^{1/p}(\sum B_i c_i)^{1/q} = (\sum a_i^p c_i)^{1/p}(\sum b_i^q c_i)^{1/q},$$

with equality if

$$\frac{a_1^p c_1}{b_1^q c_1} = \cdots = \frac{a_n^p c_n}{b_n^q c_n},$$

which is the same as the form given in the statement of the problem.

4.19. First solution. Let us use Problem 4.18 with $p = q = 2$. We have

$$1 + 2 + 4 = \sqrt{x} \cdot \frac{1}{\sqrt{x}} \cdot 1 + \sqrt{y} \cdot \frac{1}{\sqrt{y}} \cdot 2 + \sqrt{z} \cdot \frac{1}{\sqrt{z}} \cdot 4, \quad \text{or}$$

$$7 \le (x \cdot 1 + y \cdot 2 + z \cdot 4)^{1/2}\left(\frac{1}{x} + \frac{2}{y} + \frac{4}{z}\right)^{1/2},$$

with equality for $\sqrt{x}\Big/\frac{1}{\sqrt{x}} = \sqrt{y}\Big/\frac{1}{\sqrt{y}} = \sqrt{z}\Big/\frac{1}{\sqrt{z}}$, or $x = y = z$. If we square both sides of this last inequality we obtain the desired result.

Second solution. By the arithmetic-geometric means relation we have first

$$\frac{x + 2y + 4z}{7} \ge \sqrt[7]{xy^2z^4},$$

with equality only if $x = y = z$. Also,

$$\frac{\frac{1}{x} + \frac{2}{y} + \frac{4}{z}}{7} \geq \sqrt[7]{x^{-1}y^{-2}z^{-4}} ,$$

with equality only if $x = y = z$. If we multiply these two inequalities, we obtain the desired result, and, since equality holds for the same values of x,y,z, we have the condition of equality: $x = y = z$. (We remark that if we had used the inequalities

$$\frac{x + 2y + 4z}{3} \geq \sqrt[3]{x(2y)(4z)} = \sqrt[3]{8xyz}$$

$$\frac{\frac{1}{x} + \frac{2}{y} + \frac{4}{z}}{3} \geq \sqrt[3]{\frac{8}{xyz}} ,$$

we obtain that $(x + 2y + 4z)(\frac{1}{x} + \frac{2}{y} + \frac{4}{z}) \geq 9 \sqrt[3]{64} = 36$, which is true enough, but the last two inequalities have differing conditions for equality, the first being $x = 2y = 4z$, and the second $\frac{1}{x} = \frac{2}{y} = \frac{4}{z}$, which therefore cannot yield the condition $x = y = z$.

4.20. Let $c_i = 2^i$, $i = 1, \cdots, n$. Then $\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} 2^i = 2^{n+1} - 2$. We write

$$2^{n+1} - 2 = \sum c_i = \sum \sqrt{x_i} \frac{1}{\sqrt{x_i}} c_i \leq \left(\sum 2^i x_i\right)^{1/2} \left(\sum \frac{2^i}{x_i}\right)^{1/2} ,$$

so that, on squaring,

$$4(2^n - 1)^2 \leq \left(\sum_{i=1}^{n} 2^i x_i\right)\left(\sum_{i=1}^{n} \frac{2^i}{x_i}\right) ,$$

and, on dividing both sides by 4, we obtain the result, with equality only if $x_1 = x_2 = \cdots = x_n$.

Another solution parallels the second solution of Problem 4.19.

4.21. Let us use Problem 4.18 again with $p = q = 2$. We have

$$a + b + c + d \le \left(\frac{a}{c+d} + \frac{b}{d+a} + \frac{c}{a+b} + \frac{d}{b+c}\right)^{1/2} \Big(a(c+d) + b(d+a) + c(a+b) + d(b+c)\Big)^{1/2},$$

where we have used $c_1 = a$, $c_2 = b$, $c_3 = c$, $c_4 = d$. Thus we obtain the inequality

$$\frac{(a + b + c + d)^2}{[a(c+d) + b(d+a) + c(a+b) + d(b+c)]} \le S,$$

with equality only if $c + d = d + a = a + b = b + c$, or $a = b = c = d$. The value of the minimum is therefore $S_{min} = 2$.

4.22. We shall use Problem 4.18 with $c_1 = a^2$, $c_2 = b^2$, $c_3 = c^2$, and with $p = q = 2$. We have

$$\begin{aligned} 1 &= a^2 + b^2 + c^2 \\ &\le \left(\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b}\right)^{1/2} \left(a^2(b+c) + b^2(c+a) + c^2(a+b)\right)^{1/2} \end{aligned}$$

whence

$$\frac{1}{a^2(b+c) + b^2(c+a) + c^2(a+b)} \le S = \frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} .$$

Now equality holds for $b + c = c + a = a + b$, or $a = b = c$. Thus,

when $a = b = c$, $S \geq \frac{1}{6a^3}$, and $S \geq \frac{1}{6b^3}$, $S \geq \frac{1}{6c^3}$, whence $1/S \leq 6a^3$; $1/S \leq 6b^3$; $1/S \leq 6c^3$. If we add these last inequalities, we have $1/S \leq 2(a^3 + b^3 + c^3)$, whence $S \geq \frac{1}{2(a^3 + b^3 + c^3)}$, with equality only if $a = b = c = \frac{1}{\sqrt{3}}$, whence $S \geq \sqrt{3}/2$.

4.23. We use Problem 4.18 with any p and q satisfying (4.1). We have

$$1 = a^2 + b^2 + c^2 = (\frac{1}{a} \cdot a \cdot a^2 + \frac{1}{b} \cdot b \cdot b^2 + \frac{1}{c} \cdot c \cdot c^2)$$

$$\leq \left(\frac{1}{a^p} a^2 + \frac{1}{b^p} b^2 + \frac{1}{c^p} c^2\right)^{1/p} (a^q a^2 + b^q b^2 + c^q c^2)^{1/q},$$

or $(a^{2-p} + b^{2-p} + c^{2-p})^q (a^{2+q} + b^{2+q} + c^{2+q})^p \geq 1$, with equality if $a = b = c$.

4.24. The case that $1/p + 1/q = 1$ is Hölder's inequality, so that we may assume that $1/p + 1/q < 1$. The number r defined by

$$\frac{1}{r} = 1 - \frac{1}{p} - \frac{1}{q}$$

is greater than 1, so that, by Problem 4.17 extended to arbitrary p, q, r, we have

$$\frac{f^p}{p} + \frac{g^q}{q} + \frac{1^r}{r} \geq f\,g\,.$$

By (4.24) we have

$$\int_a^b f\,g\,1\;dx \leq \left(\int_a^b f^p dx\right)^{1/p} \left(\int_a^b g^q\right)^{1/q} \left(\int_a^b 1^r dx\right)^{1/r}$$

$$= (b - a)^{1/r} \left(\int_a^b f^p dx\right)^{1/p} \left(\int_a^b g^q dx\right)^{1/q},$$

with equality only if $f(x)$ and $g(x)$ are constant. The constant k turns out to be $(b - a)^{1/r}$.

4.25. The solution parallels that of Problem 3.30, where we use Hölder's inequality instead of Schwarz's inequality. Indeed, we have first that, for any M and N,

$$f^2(N) - f^2(M) = \int_M^N 2f(x)f'(x)\,dx$$

$$\leq 2\left(\int_M^N |f(x)|^p dx\right)^{1/p}\left(\int_m^N |f'(x)|^q dx\right)^{1/q}.$$

The details from this point on are essentially identical to those of the solution of Problem 3.30, and we omit them.

4.26. If we apply the results of Problem 4.18 to the identity

$$\frac{n(n+1)}{2} = 1 + 2 + 3 + \cdots + n = \sqrt{x_1}\cdot\frac{1}{\sqrt{x_1}} + 2\sqrt{x_2}\,\frac{1}{\sqrt{x_2}} + \cdots + n\sqrt{x_n}\,\frac{1}{\sqrt{x_n}},$$

where we set $c_i = i$, $a_i = \sqrt{x_i}$ and $b_i = 1/\sqrt{x_i}$, we have, for $p = q = 2$,

$$\frac{n^2(n+1)^2}{4} \leq (x_1 + 2x_2 + \cdots + nx_n)\left(\frac{1}{x_1} + \frac{2}{x_2} + \cdots \frac{n}{x_n}\right),$$

with equality for $x_1 = x_2 = \cdots = x_n$.

4.27. Use the idea of the solution of Problem 4.26, together with the identity

$$\frac{n^2(n+1)^2}{4} = 1^3 + 2^3 + \cdots + n^3$$

$$= \sum_{k=1}^{n} k^3\, x_k^{2/3}\, x_k^{-2/3},$$

and take $p = 3$, $q = 3/2$ in the inequality of Problem 4.18.

Solutions for Chapter V

5.1. If $f(x) = \log x$, then $f'(x) = 1/x$ and $f''(x) = -1/x^2$, which is negative for $x > 0$. By Theorem 5.1, $f(x)$ is concave for $0 < x < \infty$, so that the inequality is reversed in (5.3). Thus

$$\log\left(\sum_{i=1}^{n} \frac{x_i}{n}\right) \geq \sum_{i=1}^{n} \frac{1}{n} \log x_i = \log \prod_{i=1}^{n} x_i^{1/n} = \log \sqrt{x_1 \cdots x_n}\ ,$$

which is (5.18). Since $\log x$ is not linear over any interval, equality holds in (5.18) only if $x_1 = \cdots = x_n$.

5.2. We know from Problem 5.1 that $\log x$ is concave for $x > 0$. By (5.8), with the direction of inequality reversed, we have

$$\log\left(\sum_{i=1}^{n} \alpha_i x_i\right) \geq \sum_{i=1}^{n} \alpha_i \log x_i = \log \prod_{i=1}^{n} x_i^{\alpha_i}\ ,$$

which gives (5.19). Equality holds only if $x_1 = \cdots = x_n$.

5.3. In (5.19), let us set $n = 2$, $\alpha_1 = 1/p$, $\alpha_2 = 1/q$, so that $\alpha_1 + \alpha_2 = 1$. Hence, for $A_1 > 0$ and $A_2 > 0$, we have $A_1^{\alpha_1}A_2^{\alpha_2} \leq \alpha_1 A_1 + \alpha_2 A_2$, with equality only if $A_1 = A_2$. If we now set $A_1 = x^p$ and $A_2 = y^q$, we have (5.20) with equality only if $x^p = y^q$.

5.4. In (5.19), we set $\alpha_i = 1/p_i$, so that $\sum_{i=1}^{n}\alpha_i = 1$. If we set $A_i = x_i^{p_i}$ and follow the steps of the solution of Problem 5.3, we obtain (5.21), with equality only if

$$x_1^{p_1} = x_2^{p_2} = \cdots = x_n^{p_n}.$$

5.5. For any $\alpha > 0$, $\beta > 0$, we have immediately from the convexity of $f(x) = x \log x$ (verify that $f''(x)$ is positive for $x > 0$) that

$$\frac{\alpha+\beta}{2}\log\left(\frac{\alpha+\beta}{2}\right) \le \frac{\alpha}{2}\log\alpha + \frac{\beta}{2}\log\beta,$$

whence $\left(\frac{\alpha+\beta}{2}\right)^{\frac{\alpha+\beta}{2}} \le (\alpha^\alpha\beta^\beta)^{1/2}$. Since x log x is linear nowhere, equality holds only if $\alpha = \beta$.

5.6. The fact that x log x is convex (Problem 5.5) assures us that, if $\Sigma p_i = 1$, $p_i \geq 0$,

$$\begin{aligned}(p_1x_1 + \cdots + p_nx_n)&\log(p_1x_1 + \cdots + p_nx_n)\\ &\le p_1x_1 \log x_1 + \cdots + p_nx_n \log x_n\\ &= \log\left(x_1^{p_1x_1} \cdots x_n^{p_nx_n}\right),\end{aligned}$$

which is (5.22). Equality holds only for $x_1 = \cdots = x_n$.

5.7. Observe that the function $-\sqrt{x}\log x$ is convex for $x > 1$. Equality holds only if $\alpha = \beta$.

5.8. If we play a bit with the given inequality, we have, since $\sin x_i > 0$, that

$$\log\sin x_1 + \cdots + \log\sin x_n \le n\log\sin\left(\frac{x_1 + \cdots + x_n}{n}\right),$$

or

$$\frac{1}{n}\sum_{k=1}^{n}\log\sin x_k \le \log\sin\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right).$$

This is the form (5.3) of Jensen's inequality for a concave function, in this case, f(x) = log sin x. After verifying that the function log sin x is indeed concave, we may reverse the steps to obtain the required inequality. Equality holds only for $x_1 = \cdots = x_n$.

5.9. If $f(x)$ is twice differentiable, then $F(x) = \log f(x)$ is concave on I because $F'' = (ff'' - f'^2)/f^2$ cannot be positive because f'' cannot be positive. However, the concavity of $F(x)$ may be proved in a more satisfying way as follows. If x_1 and x_2 are any two points on I, then

$$\begin{aligned} F\left(\frac{x_1 + x_2}{2}\right) &= \log f\left(\frac{x_1 + x_2}{2}\right) \\ &\geq \log\left[\frac{f(x_1) + f(x_2)}{2}\right] && \text{(f is concave)} \\ &\geq \frac{1}{2}\log f(x_1) + \frac{1}{2}\log f(x_2) && \text{(log x is concave)} \\ &= \frac{1}{2}F(x_1) + \frac{1}{2}F(x_2). \end{aligned}$$

Thus, if $x_1, x_2, \cdots, x_n$ are points of I, we have

$$\log f\left(\frac{x_1 + \cdots + x_n}{n}\right) \geq \frac{1}{n}\log f(x_1) + \cdots + \frac{1}{n}\log f(x_n),$$

or

$$[f(x_1)f(x_2)\ \cdots\ f(x_n)]^{1/n} \leq f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right),$$

as desired. Equality holds only for $x_1 = \cdots = x_n$, for if $F(x) = \log f(x)$ were linear this would imply that $f(x)$ is exponential and therefore convex, contradicting the assumption that $f(x)$ is concave.

Remark. By using the stronger form (5.8) of Jensen's inequality for a concave, rather than convex, function, we have, for positive numbers $\alpha_1, \cdots, \alpha_n$ with $\Sigma\alpha_i = 1$,

$$\log f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \geq \sum_{i=1}^{n} \alpha_i \log f(x_i),$$

or

$$[f(x_1)]^{\alpha_1} \cdots [f(x_n)]^{\alpha_n} \le f\left(\sum_{i=1}^{n} \alpha_i x_i\right),$$

which involves the weighted arithmetic and geometric means. Equality holds only when $x_1 = \cdots = x_n$.

5.10. Because $p - 1 > 0$, the function $f(x) = x^{p-1}$, $f(0) = 0$, is increasing for all $x > 0$ and hence has an inverse function $g(y)$, which we may calculate from the relation $y = x^{p-1}$, namely, $x = y^{1/(p-1)}$, or because $1/p + 1/q = 1$, $x = y^{q-1}$, which is an increasing function $y > 0$ and vanishes at $y = 0$. Now

$$A_1 = \int_0^a x^{p-1}dx = \frac{a^p}{p}, \qquad A_2 = \int_0^b y^{q-1}dy = \frac{b^q}{q},$$

and the figure below shows that the area of the rectangle with sides a and b cannot exceed the sum of the areas A_1 and A_2. Equality holds only if $b = f(a) = a^{p-1}$, or, what is the same thing, if $a^p = b^q$.

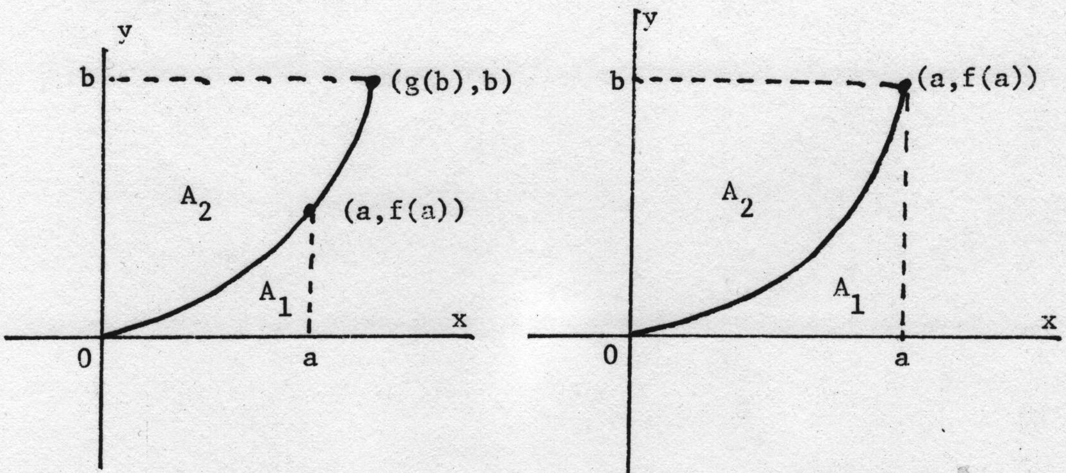


Fig. I Fig. II

5.11. An intuitive geometrical proof is clear if we modify the solution given to Problem 5.10 and use the figures in that solution. However, we give a solution that is applicable in a wider context; for example, in the case that the functions $f(x)$ and $g(x)$ are not continuous, the solution may be modified in view of the nature of the discontinuities.

If we set $y = f(x)$ and $x = g(y)$, then the form of $A_1 + A_2$ suggests a line integral

$$\int_{(0,0)}^{(\alpha,\beta)} y\,dx + x\,dy \;,$$

with the upper limit to be determined. Since y is positive and $g(b) > a$, we have

$$\begin{aligned} A_1 + A_2 &= \int_{(0,0)}^{(g(b),b)} y\,dx + x\,dy - \int_{(a,f(a))}^{(g(b),b)} y\,dx \\ &\geq \int_{(0,0)}^{(g(b),b)} y\,dx + x\,dy = \int_{(0,0)}^{(g(b),b)} d(xy) \\ &= xy\Big|_{(0,0)}^{(g(b),b)} = g(b)\cdot b \\ &\geq ab. \end{aligned}$$

Here, we have assumed that, as we traverse the curve from the origin, we meet the point $(a,f(a))$ before the point $(g(b),b)$, as is the case in Figure I above. The other possibility, that we meet the point $(g(b),b)$ before meeting the point $(a,f(a))$, is not illustrated, but in this case we have

$$\begin{aligned} A_1 + A_2 &= \int_{(0,0)}^{(a,f(a))} d(xy) - \int_{(g(b),b)}^{(a,f(a))} x\,dy \\ &\geq \int_{(0,0)}^{(a,f(a))} d(xy) = af(a) \\ &\geq ab, \end{aligned}$$

which gives the same result. The case of equality holds when $b = f(a)$, or $a = g(b)$, which is shown in Figure II.

5.12. Using the result of the preceding problem, we have

$$ax \leq \int_0^a \sin t \, dt + \int_0^x \sin^{-1} t \, dt$$

$$= (-\cos a + 1) + x \sin^{-1} x + \sqrt{1 - x^2} - 1,$$

or

$$\cos a + ax \leq x \sin^{-1} x + \sqrt{1 - x^2} \; ,$$

and the result follows by setting $a = \pi/2$. Equality holds only for $x = 1$.

5.13. Using the result of Problem 5.11, we have, since $f(x) = x^5 + x$ and $g(y)$ are increasing,

$$ab \leq \int_0^a (x^5 + x)dx + \int_0^b g(y)dy,$$

with equality for $b = a^5 + a$. To evaluate $\int_0^2 g(y)dy$, we must have equality, i.e., $a^5 + a = 2$, or $a = 1$; this is the only solution for $a > 0$. Hence

$$1.2 = \int_0^1 (x^5 + x)dx + \int_0^2 g(y)dy,$$

or

$$\int_0^2 g(y)dy = 4/3.$$

5.14. Using the form of (5.3) for concave functions (with equality reversed), we have

$$\log\left(1 + \sum_{k=1}^{n} \frac{a_k}{n}\right) \geq \frac{1}{n} \sum_{k=1}^{n} \log(1 + a_k)$$

or

$$(1 + A)^n \geq \prod_{k=1}^{n} (1 + a_k),$$

with equality only if $a_1 = \cdots = a_n$.

5.15. That $f(x) = \log(1 + e^x)$ is convex follows from the fact that $f''(x) = e^x(1 + e^x)^{-2}$ is positive for all x. Thus for any n real numbers $x_1, \cdots, x_n$, it follows from (5.3) that

$$\left(1 + e^{\frac{x_1+\cdots+x_n}{n}}\right)^n \leq \prod_{k=1}^{n} \left(1 + e^{x_k}\right),$$

or

$$\left(1 + \left[e^{x_1} \cdots e^{x_n}\right]^{1/n}\right)^n \leq \prod_{k=1}^{n} \left(1 + e^{x_k}\right).$$

Since the a_k are positive, we may set $\log a_k = x_k$, or $a_k = e^{x_k}$, whence

$$(1 + (a_1 a_2 \cdots a_n)^{1/n})^n \leq \prod_{k=1}^{n} (1 + a_k),$$

which is what we want to show. Equality holds only if $x_1 = \cdots = x_n$, or, what is the same thing, $a_1 = \cdots = a_n$.

5.16. The last displayed inequality in the solution of Problem 5.15 may be written as

$$1 + (a_1 a_2 \cdots a_n)^{1/n} \leq \prod_{k=1}^{n} (1 + a_k)^{1/n}.$$

If, in this inequality, we replace a_k by a_k/b_k, we have

$$1 + \prod_{k=1}^{n} a_k^{1/n} \Big/ \prod_{k=1}^{n} b_k^{1/n} \leq \prod_{k=1}^{n} \left(1 + \frac{a_k}{b_k}\right)^{1/n},$$

and, if we multiply both sides by $\Pi b_k^{1/n}$, we obtain the desired inequality. Equality holds if $a_1/b_1 = \cdots = a_n/b_n$.

We remark that it is quite simple to show that

$$\prod_{k=1}^{n} a_k^{1/n} + \prod_{k=1}^{n} b_k^{1/n} + \prod_{k=1}^{n} c_k^{1/n} \leq \prod_{k=1}^{n} (a_k + b_k + c_k)^{1/n}$$

by applying the original result of this problem to the product

$$\prod_{k=1}^{n} ([a_k + b_k] + c_k)^{1/n}.$$

5.17. Raise both sides of the inequality in Problem 5.16 to the n-th power, and recognize the geometric means g_a and g_b. In accordance with the remark at the end of the solution of Problem 5.16, we also have

$$(g_a + g_b + g_c)^n \leq \prod_{k=1}^{n} (a_k + b_k + c_k),$$

where g_c is the geometric mean of the positive numbers $c_1, \cdots, c_n$.

5.18. The method of solution of Problem 5.11 may be adapted here. If we set, as before, $A_1 = \int_0^a p(x,f(x))dx$ and $A_2 = \int_0^b q(g(y),y)dy$, then we may write $A_1 + A_2$ in terms of the line integrals

$$A_1 + A_2 = \int_{(0,0)}^{(g(b),b)} p(x,y)dx + q(x,y)dy - \int_{(a,f(a))}^{(g(b),b)} p(x,y)dx$$

or

$$A_1 + A_2 = \int_{(0,0)}^{(a,f(a))} p(x,y)dx + q(x,y)dy - \int_{(g(b),b)}^{(a,f(a))} q(x,y)dy,$$

according as $(a,f(a))$ or $(g(b),b)$ is encountered first as we traverse the path of integration starting at the origin. Since $p(x,y)$ and $q(x,y)$ are both positive, we have, in both cases above, that

$$A_1 + A_2 \geq \int_{(0,0)}^{(g(b),b)} p(x,y)dx + q(x,y)dy$$

$$= F(x,y)\Big|_{(0,0)}^{(g(b),b)} = F(g(b),b)$$

$$\geq F(a,b),$$

and

$$A_1 + A_2 \geq \int_{(0,0)}^{(a,f(a))} p(x,y)dx + q(x,y)dy$$

$$= F(x,y)\Big|_{(0,0)}^{(a,f(a))} = F(a,f(a))$$

$$\geq F(a,b).$$

Equality occurs only if $b = f(a)$, or $a = g(b)$.

5.19. If we rotate the curve $y = f(x)$ about the x-axis, the volume of the solid of rotation between $x = 0$ and $x = a$ is given by $V_x = \int_0^a \pi y^2 dx$ and $\bar{x}\, V_x = \int_0^a \pi xy^2 dx$, where $\bar{x}$ is the distance from $(0,0)$ to the centroid of the solid along the x-axis. Similarly, $V_y = \int_0^b \pi x^2 dy$ and $\bar{y}\, V_y = \int_0^b \pi yx^2 dy$. By the result of Problem 5.18, we have

$$\bar{x}\, V_x + \bar{y}\, V_y = \int_0^a \pi xy^2 dx + \int_0^b \pi x^2 y\, dy \geq \frac{\pi}{2} a^2 b^2,$$

for the function $F(x,y) = \frac{\pi}{2} x^2 y^2$ has the property that $dF = \pi xy^2 dx + \pi x^2 y\, dy$. Equality holds when $b = f(a)$, so that, in this case,

$$\bar{x}\, V_x + \bar{y}\, V_y = \frac{\pi}{2}\, a^2 [f(a)]^2.$$

5.20. (a) The function cos x is concave on $0 < x < \pi/2$, so that we must restrict the angles x_1, x_2, x_3 to be less than $\pi/2$, so that the triangle is acute. By (5.3) we have

$$\frac{1}{3}(\cos x_1 + \cos x_2 + \cos x_3) \le \cos \frac{x_1 + x_2 + x_3}{3} = 1/2,$$

with equality if $x_1 = x_2 = x_3 = \pi/3$. Hence $\cos x_1 + \cos x_2 +$ $+ \cos x_3$ assumes a maximum value 3/2 for the equilateral triangle.

(b) As in the preceding problem, we must restrict ourselves to an acute angle because cos x is concave only over $0 < x < \pi/2$. By Problem 5.9 we have

$$\cos x_1 \cos x_2 \cos x_3 \le \cos^3\left(\frac{x_1 + x_2 + x_3}{3}\right) = 1/8,$$

with equality, and hence the maximum value of the expression, occurring for $x_1 = x_2 = x_3$, again for the equilateral triangle.

(c) Again, we must restrict ourselves to the interval $0 < x < \pi/2$, where both cot x and log cot x are convex (this fact should be verified; use Theorem 5.1). We have, by (5.3), that

$$\log \cot x_1 + \log \cot x_2 + \log \cot x_3 \le 3 \log \cot \frac{x_1 + x_2 + x_3}{3},$$

or
$$\log(\cot x_1 \cot x_2 \cot x_3) \le 3 \log \cot \frac{\;}{3},$$

or
$$\cot x_1 \cot x_2 \cot x_3 \le (\cot \tfrac{\;}{3})^3 = 1/3\sqrt{3},$$

with equality for $x_1 = x_2 = x_3$.

We observe that the maximum of the given expression occurs whenever the reciprocal $\tan x_1 \tan x_2 \tan x_3$ assumes its minimum, which is $3\sqrt{3}$.

(d) The function sec x is convex on $0 < x < \pi/2$, so that, by (5.3), we have

$$\sec\left(\frac{x_1 + x_2 + x_3}{3}\right) \leq \frac{1}{3}(\sec x_1 + \sec x_2 + \sec x_3),$$

whence

$$\sec x_1 + \sec x_2 + \sec x_3 \geq 3 \sec \pi/3 = 6,$$

with equality for the equilateral triangle; the minimum value of the given expression is then 6.

(e) The function tan x/2 is convex on $0 < x < \pi$, so that, by (5.3)

$$\tan\left(\frac{x_1 + x_2 + x_3}{6}\right) \leq \frac{1}{3}(\tan x_1/2 + \tan x_2/2 + \tan x_3/3)$$

or

$$\tan \frac{x_1}{2} + \tan \frac{x_2}{2} + \tan \frac{x_3}{3} \geq 3 \tan \pi/6 = 3 / \sqrt{3}\ ,$$

so that the minimum $3 / \sqrt{3}$ is achieved in the case of the equilateral triangle.

5.21. If we set $y_1 = f(x_1)$, $y_2 = f(x_2)$, $y_3 = f(x_3)$, and recall from analytic geometry that the formula for the area of a triangle

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

will be positive if, when we traverse the edges from (x_1,y_1) to (x_2,y_2) to (x_3,y_3), the interior of the triangle is to our left; otherwise, A will be negative. We see from Figure 5.5 that this is precisely the distinction between convex and concave functions.

To prove the result algebraically, we note that the condition is equivalent to

$$0 < \begin{vmatrix} x_1 & y_1 & 1 \\ x_2-x_1 & y_2-y_1 & 0 \\ x_3-x_1 & y_3-y_1 & 0 \end{vmatrix} = (x_2-x_1)(y_3-y_1) - (x_3-x_1)(y_2-y_1),$$

and to

$$0 < \begin{vmatrix} x_1-x_2 & y_1-y_2 & 0 \\ x_2 & y_2 & 1 \\ x_3-x_2 & y_3-y_2 & 0 \end{vmatrix} = (x_2-x_1)(y_3-y_2) - (x_3-x_2)(y_2-y_1),$$

and that these two inequalities give us (5.24), which in turn gives us (5.23). The steps preceding (5.23) are reversible, so that the condition is both necessary and sufficient.

5.22. Let x and x_o lie in $a < x < b$, $x > x_o$. We have already seen that

$$f'_+(x) \leq \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

for $\Delta x > 0$. Since $f(x)$ is continuous and $f'_+(x)$ is monotonically increasing, we may take the limit of this inequality as $x \to x_o$, $x > x_o$, to obtain

$$\lim_{\substack{x \to x_o \\ x > x_o}} f'_+(x) \leq \frac{f(x_o + \Delta x) - f(x)}{x}$$

Thus, as $\Delta x \to 0$, $\Delta x > 0$, we have

$$\lim_{\substack{x \to x_o \\ x > x_o}} f'_+(x) \leq f'_+(x_o).$$

But we know already that $f'_+(x_o) \leq f'_+(x)$ for $x_o < x$, whence

$$\lim_{\substack{x \to x_o \\ x > x_o}} f'_+(x) \geq f'_+(x_o).$$

The last two inequalities show that

$$\lim_{\substack{x \to x_o \\ x > x_o}} f'_+(x) = f'_+(x_o),$$

which is what we want. The result that

$$\lim_{\substack{x \to x_o \\ x < x_o}} f'_-(x) = f'_-(x_o)$$

is handled analogously.

5.23. Let $j_x = f'_+(x) - f'_-(x)$; if $j_x = 0$ for some value x in I, then $f'(x)$ exists at the point x. If $j_x > 0$, we define an interval J_x on the y-axis by J_x: $f'_-(x) < y < f'_+(x)$. Note that if $x_1 \neq x_2$ and if $j_{x_1} > 0$ and $j_{x_2} > 0$, the two intervals J_{x_1} and J_{x_2} have no points in common, for both $f'_-(x)$ and $f'_+(x)$ are non-decreasing functions of x. Since each interval J_x contains a rational point on the y-axis, there cannot be more intervals J_x than there are rational points, that is, the set of points x such that $j_x > 0$ is denumerable.

We remark that the proof uses the fact that $f'_+(x)$ and $f'_-(x)$ are non-decreasing. Actually, it can be proved by elementary means that a function $F(x)$ possessing right-hand and left-hand derivatives at every point of $a < x < b$ has a derivative $F'(x)$ at all points of the interval except possibly at a denumerable set of points.

5.24. We set $m = \frac{1}{\pi r^2} \iint_{D_r} f(x,y)dxdy$ and call $g = f - m$, so that $\iint_{D_r} g\, dxdy = 0$.

Let us use (5.26) in the following calculation:

$$\begin{aligned}
\frac{1}{\pi r^2} \iint_{D_r} \log f(x,y)dxdy &= \frac{1}{\pi r^2} \iint_{D_r} \log(m + g)dxdy \\
&= \log m + \frac{1}{\pi r^2} \iint_{D_r} \log(1 + \frac{g}{m})dxdy \\
&\leq \log m + \frac{1}{\pi r^2} \iint_{D_r} \frac{g}{m}\, dxdy = \log m,
\end{aligned}$$

which proves the result. Equality holds only if $g = 0$, that is, only if f is constant.